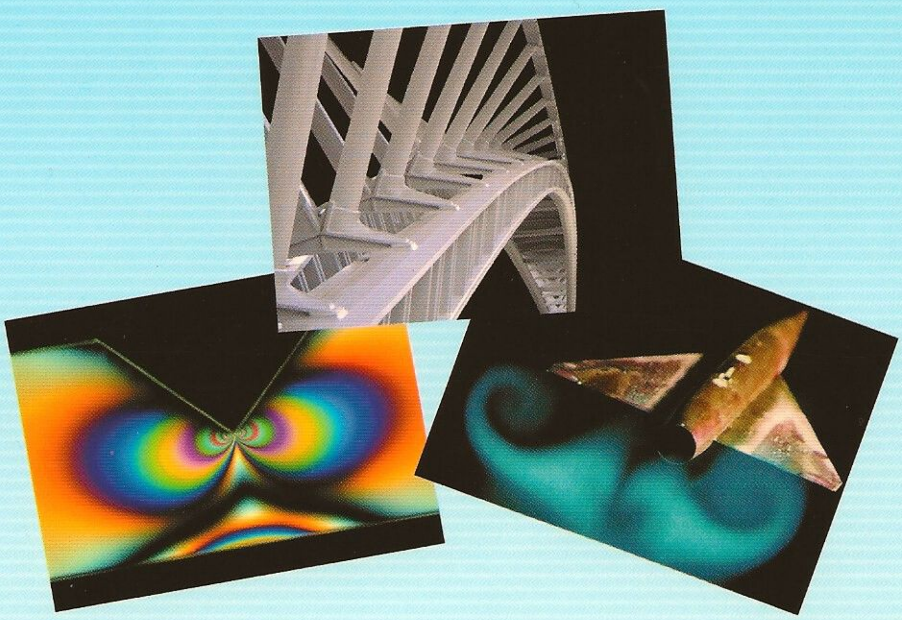
Jean Salençon

Continuum Mechanics
Short Reader



LES ÉDITIONS DE L'ÉCOLE POLYTECHNIQUE

Continuum Mechanics
Short Reader

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Translated by Stephen Lyle

Direction des Études - 91128 Palaiseau Cedex

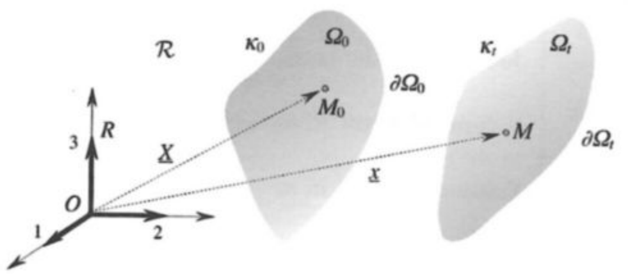
Contents

1. Modelling the Continuum

Notation	Meaning
$\underline{X}$	Position vector in κ_0
$\underline{x}$	Position vector in κ_t
$\underline{\phi}(\underline{X}, t)$	Bijection of Ω_0 on Ω_t
$J(\underline{X}, t)$	Jacobian of the transformation
$\mathrm{d}\Omega_0$	Volume element in κ_0
$\mathrm{d}\Omega_t$	Volume element in κ_t
$\underline{U}(\underline{X}, t)$	Velocity in the Lagrangian description
$\underline{U}_t(\underline{x}, t)$	Velocity in the Eulerian description
$\mathcal{B}$	Value of a physical quantity
$B(\underline{X}, t)$	Physical quantity associated with a particle in the Lagrangian description
$b(\underline{x}, t)$	Quantity at a geometrical point in the Eulerian description

• The concept of a continuous medium is a *macroscopic physical model* arising from common experience. Its relevance depends on the problem at hand and the scale of the phenomena to be modelled.

In the mathematical formulation of this concept, a mechanical system is represented by a volume made up, on the differential level, of ***particles***. The geometrical state of these particles, rather like that of a material point, is completely characterised by giving its position in a reference frame $\mathcal{R}$. The intuitive perception of continuity appeals to the evolution of the system, during which particles initially close together remain so.



• In order to specify such a model, the *Lagrangian description* identifies particles by their position in a *configuration* of the system taken as *reference*. The motion is described by defining the position of each particle during the evolution. In other words, we specify its *pathline*, parametrised by *time* in the form

$$\underline{x} = \underline{\phi}(\underline{X}, t) .$$

All physical quantities are defined in the same manner. Hence, for a physical quantity associated with a particle,

$$\mathcal{B} = B(\underline{X}, t) .$$

The continuity of the medium is expressed by the *spatial and temporal continuity* of the correspondence between the initial position of the particle and its current position. Continuous differentiability conditions with respect to the variables $\underline{X}$ and t are also imposed:

$$\begin{aligned} &\underline{\phi} \text{ bijective, of class } C^1 \text{ or } C^2 \text{, and likewise for } \underline{\phi}^{-1} , \\ &B \text{ of class } C^1 \text{ or } C^2 , \end{aligned}$$

from which

$$0 < J(\underline{X}, t) = \frac{D(x^1, x^2, x^3)}{D(X^1, X^2, X^3)} < +\infty \,,$$
$$\mathrm{d}\Omega_t = J(\underline{X}, t)\, \mathrm{d}\Omega_0 \,,$$

and the velocity of the particle is

$$\underline{U}(\underline{X}, t) = \frac{\partial \underline{\phi}(\underline{X}, t)}{\partial t} \,.$$

Experimental *validation* of the model shows that these regularity hypotheses can be weakened in the sense that they need only be imposed in a piecewise manner.

- The *Eulerian description* takes an incremental standpoint, defining the motion of the system by giving, at each instant of time, the velocity field of the particles. This formulation is made in the *current configuration* and the spatial variables occurring in it have a purely geometrical meaning,

$$\underline{U} = \underline{U}_t(\underline{x}, t) \,.$$

In the Eulerian description, the notion of continuity corresponds to the piecewise continuity and continuous differentiability of the velocity field with respect to space and time.

All physical quantities are defined at each time in the current configuration as a function of the particle coordinates, viz.,

$$\mathcal{B} = b(\underline{x}, t) \,.$$

- The Lagrangian and Eulerian descriptions are equivalent: pathlines

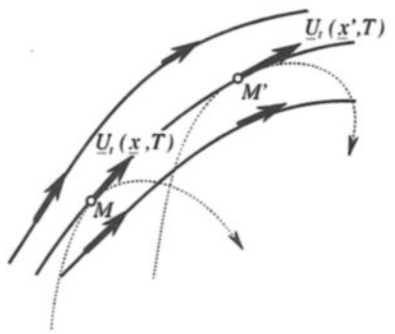
$$\left.\begin{array}{l} \dfrac{\mathrm{d}\underline{x}}{\mathrm{d}t} = \underline{U}_t(\underline{x}, t) \\ \underline{x}|_{t=0} = \underline{X} \end{array}\right\} \longrightarrow \underline{x} = \underline{\phi}(\underline{X}, t)$$

and

$$B(\underline{X}, t) = b\big(\underline{\phi}(\underline{X}, t), t\big) \,.$$

In a frame of reference $\mathcal{R}$, the streamlines at time T are the envelope curves of the velocity field, defined by

$$\frac{\mathrm{d}x^1}{U_t^1(\underline{x}, T)} = \frac{\mathrm{d}x^2}{U_t^2(\underline{x}, T)} = \frac{\mathrm{d}x^3}{U_t^3(\underline{x}, T)} \,.$$



• The motion is *steady* in the frame $\mathcal{R}$ if its Eulerian description is independent of t, so that

$$\underline{U}_t(\underline{x}, t) \equiv \underline{U}(\underline{x}) \,.$$

The streamlines are then independent of time and identical to the pathlines of the particles.

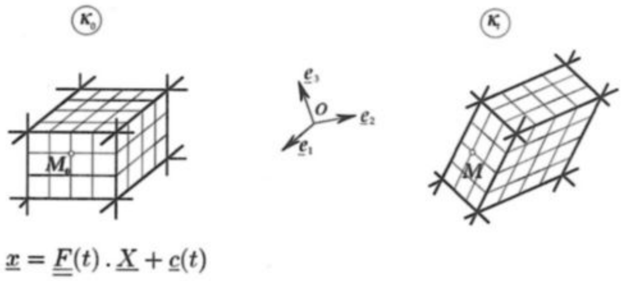
2. Deformation

Notation	Meaning
$\underline{\underline{F}}(\underline{X}, t)$	Gradient of a transformation
$\underline{\mathrm{d}M}_0\,,\ \mathrm{d}s_0$	Line element in κ_0, $\mathrm{d}s_0 = \lvert\underline{\mathrm{d}M}_0\rvert$
$\underline{\mathrm{d}M}\,,\ \mathrm{d}s$	Line element in κ_t, $\mathrm{d}s = \lvert\underline{\mathrm{d}M}\rvert$
$\underline{\mathrm{d}A} = \underline{N}\,\mathrm{d}A$	Surface element in κ_0
$\underline{\mathrm{d}a} = \underline{n}\,\mathrm{d}a$	Surface element in κ_t
$\underline{\underline{C}}(\underline{X}, t)$	Expansion tensor
$\underline{\underline{e}}(\underline{X}, t)$	Green–Lagrange strain tensor
$\underline{\xi}(\underline{X}, t)$	Displacement
$\underline{\underline{\varepsilon}}(\underline{X}, t)$	Linearised strain tensor
∇	Gradient in κ_0
grad	Gradient in κ_t

• In the Lagrangian representation, the comparison between the current and initial configurations, without any reference to the intervening history of the system in question, introduces two key concepts on the geometrical level, namely, transport and deformation.

• The concept of *convective transport*, most simply exemplified for the material point, expresses the correspondence between current and initial positions of material elements.

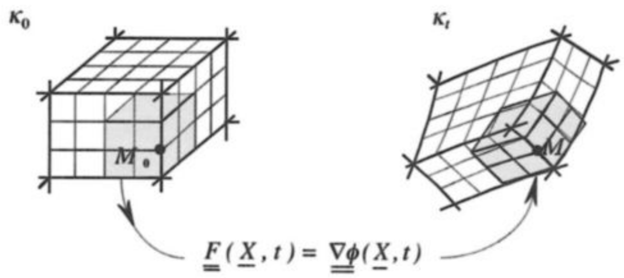
• The special case in which the system undergoes a *homogeneous transformation*, that is, one which is the same everywhere, allows us to ignore local effects to begin with. The transformation is the linear mapping defined by the tensor $\underline{\underline{F}}(t)$.



• For the general case in which the transformation of the system is arbitrary, we appeal to the notion of *homogeneous tangent map*. This is defined by the gradient tensor of the vector function expressing convective transport of the material point. We call this the *gradient of the transformation*,

$$\underline{\underline{F}}(\underline{X}, t) = \underline{\underline{\nabla \phi}}(\underline{X}, t) .$$

The idea behind this is that, locally, the transformation of the infinitesimal element is quasi-homogeneous at each point.



The transport equations for a material vector, surface or volume involve the Euclidean tensor associated with this linear mapping:

$$\begin{aligned}\underline{\mathrm{d}M} &= \underline{\underline{F}}(\underline{X},t)\,.\,\underline{\mathrm{d}M_0}\;,\\ \mathrm{d}\Omega_t &= \mathrm{d}\Omega_0\,\det\big(\underline{\underline{F}}(\underline{X},t)\big) = J(\underline{X},t)\,\mathrm{d}\Omega_0\;,\\ \underline{\mathrm{d}a} &= J(\underline{X},t)\,{}^t\underline{\underline{F}}^{-1}(\underline{X},t)\,.\,\underline{\mathrm{d}A}\;.\end{aligned}$$

• The concept of *deformation* expresses locally the extent to which the transformation suffered by the system from one configuration to another differs from a *direct isometry*. Indeed, it measures the local change of shape. To this end, we introduce the *expansion tensor*

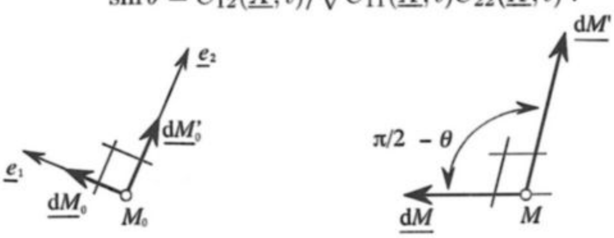
$$\underline{\underline{C}}(\underline{X},t) = {}^t\underline{\underline{F}}(\underline{X},t)\,.\,\underline{\underline{F}}(\underline{X},t)$$

and the Green–Lagrange *strain tensor*

$$\underline{\underline{e}}(\underline{X},t) = \frac{1}{2}\big(\underline{\underline{C}}(\underline{X},t) - \underline{\underline{1}}\big)\;.$$

These enable us to express changes in length and angular variations

$$\begin{aligned}\underline{\mathrm{d}M}\,.\,\underline{\mathrm{d}M'} &= \underline{\mathrm{d}M_0}\,.\,\underline{\underline{C}}(\underline{X},t)\,.\,\underline{\mathrm{d}M_0'}\;,\\ \mathrm{d}s^2 - \mathrm{d}s_0^2 &= 2\,\underline{\mathrm{d}M_0}\,.\,\underline{\underline{e}}(\underline{X},t)\,.\,\underline{\mathrm{d}M_0}\;,\\ \sin\theta &= C_{12}(\underline{X},t)/\sqrt{C_{11}(\underline{X},t)C_{22}(\underline{X},t)}\;.\end{aligned}$$



• In the orthonormal basis of their common *principal axes* in κ_0, $\underline{\underline{C}}(\underline{X},t)$ and $\underline{\underline{e}}(\underline{X},t)$ assume diagonal form. Consequently,

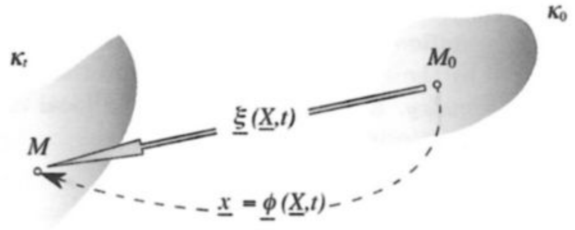
$$\underline{\underline{C}}(\underline{X},t) = \lambda_1^2(\underline{X},t)\,\underline{e}_1\otimes\underline{e}_1 + \lambda_2^2(\underline{X},t)\,\underline{e}_2\otimes\underline{e}_2 + \lambda_3^2(\underline{X},t)\,\underline{e}_3\otimes\underline{e}_3$$

($\lambda_i(\underline{X},t) > 0$, $i = 1,2,3$, principal stretches).

The *orthogonal triad* of these principal axes in κ_0 is convectively transported in κ_t along orthogonal directions (characterising property).

- We define the displacement

$$\underline{\xi}(\underline{X},t) = \underline{\phi}(\underline{X},t) - \underline{X} .$$



- When the transformation is *infinitesimal*, equations can be simplified by *linearisation*, giving

$$\|\underline{\underline{\nabla\xi}}(\underline{X},t)\| \ll 1 ,$$
$$\underline{\underline{e}}(\underline{X},t) \simeq \underline{\underline{\varepsilon}}(\underline{X},t) = \frac{1}{2}\big(\,\underline{\underline{\nabla\xi}}(\underline{X},t) + {}^t\underline{\underline{\nabla\xi}}(\underline{X},t)\,\big) ,$$
$$\underline{\underline{\varepsilon}}(\underline{X},t) \simeq \frac{1}{2}\big(\,\underline{\underline{\mathrm{grad}\ \xi}}(\underline{x},t) + {}^t\underline{\underline{\mathrm{grad}\ \xi}}(\underline{x},t)\,\big), \quad \underline{x} = \underline{\phi}(\underline{X},t) ,$$
$$\mathrm{d}\Omega_t \simeq \mathrm{d}\Omega_0\,\big(1 + \mathrm{div}\,\underline{\xi}(\underline{X},t)\,\big) ,$$
$$(\mathrm{d}s - \mathrm{d}s_0)/\mathrm{d}s_0 = \varepsilon_{11}(\underline{X},t) ,$$
$$\theta = 2\,\varepsilon_{12}(\underline{X},t) .$$

- The deformations are generated by external excitations of mechanical, thermal, or hygrometric origin, or caused by chemical processes, structural reorganisation, and so on. The question of their *geometrical compatibility* thus arises: are these deformations compatible with the continuity of the medium?

The necessary conditions for geometrical compatibility can be written, if $\|\underline{\underline{\nabla\xi}}\| \ll 1$,

$$\varepsilon_{ij,k\ell} + \varepsilon_{k\ell,ij} - \varepsilon_{ik,j\ell} - \varepsilon_{j\ell,ik} = 0\ ,\ i,j,k,\ell = 1,2,3\ ,$$

in orthonormal Cartesian coordinates. These six conditions are also sufficient if the domain of definition of the field $\underline{\underline{\varepsilon}}$ is simply connected. Otherwise, closure conditions must be satisfied. In every case, we must check that the displacement field solution is compatible with whatever boundary conditions are imposed upon it.

3. Kinematics

Notation	Meaning
$\underline{\underline{\operatorname{grad} U}}(\underline{x}, t)$	Eulerian gradient of the velocity field
$\underline{\underline{d}}(\underline{x}, t)$	(Eulerian) strain rate
$\dot{\theta}$	Angular distortion rate
$\underline{\underline{\Omega}}(\underline{x}, t)$	Spin tensor
$\underline{\Omega}(\underline{x}, t)$	Spin vector
$\frac{\mathrm{d}}{\mathrm{d}t}$ or $\dot{}$	Symbols for the material derivative
$\underline{W}$	Propagation velocity
$[\![\ ]\!]$	Symbol for discontinuity
$\rho(\underline{x}, t)$	Mass per unit volume in κ_t
$\rho_0(\underline{X})$	Mass per unit volume in κ_0

The evolution of a system can be described from an incremental point of view either within the Lagrangian or the Eulerian formulation.

• In the Lagrangian description, the time derivative is identified as the *material derivative*, which follows the evolution of a quantity associated with a particle, a discrete set of particles, or a material element. On a geometrical level, the kinematics of the continuum follow directly from the study of convective transport, transformation and deformation between an initial reference configuration and the current configuration.

The Lagrangian strain rate is naturally $\dot{\underline{\underline{e}}}\,(\underline{X},t)$, leading to

$$\dot{\widehat{\underline{\mathrm{d}M}\,.\,\underline{\mathrm{d}M'}}} = 2\,\underline{\mathrm{d}M_0}\,.\,\dot{\underline{\underline{e}}}\,(\underline{X},t)\,.\,\underline{\mathrm{d}M'_0}\;.$$

• The incremental view finds its full mathematical coherence in the Eulerian description, which it in fact inspires. At each time, the coming infinitesimal evolution is defined in the current configuration. On the geometrical level, the motion being given by the velocity field

$$\underline{U} = \underline{U}(\underline{x},t)\;,$$

it is the gradient of this field in the current configuration which defines the infinitesimal transformation locally,

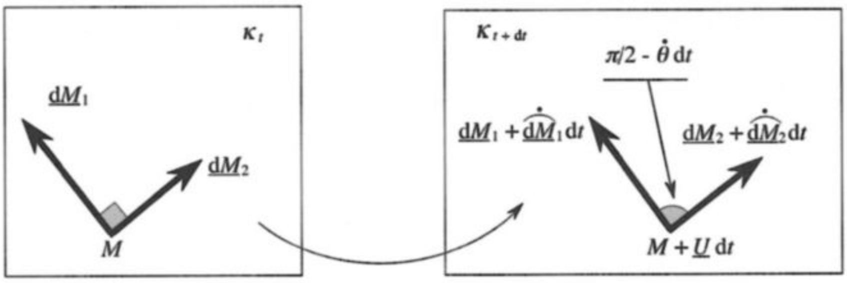
$$\begin{aligned}
\dot{\widehat{\underline{\mathrm{d}M}}} &= \underline{\underline{\operatorname{grad} U}}(\underline{x},t)\,.\,\underline{\mathrm{d}M}\;,\\
\dot{\widehat{\mathrm{d}\Omega_t}} &= \operatorname{div}\underline{U}(\underline{x},t)\,\mathrm{d}\Omega_t\;.
\end{aligned}$$

• The (Eulerian) *strain rate* tensor, defined as the symmetric part of this gradient,

$$\underline{\underline{d}}(\underline{x},t) = \frac{1}{2}\big(\underline{\underline{\operatorname{grad} U}}(\underline{x},t) + {}^t\underline{\underline{\operatorname{grad} U}}(\underline{x},t)\big)\;,$$

characterises the evolution of the metric, that is, of the deformation, expressed relative to the current configuration, which plays here the role of reference configuration at each instant of time:

$$\begin{aligned}
\dot{\widehat{\underline{\mathrm{d}M}\,.\,\underline{\mathrm{d}M'}}} &= 2\,\underline{\mathrm{d}M}\,.\,\underline{\underline{d}}(\underline{x},t)\,.\,\underline{\mathrm{d}M'}\;,\\
\frac{\dot{\widehat{\mathrm{d}s}}}{\mathrm{d}s} &= d_{11}(\underline{x},t)\;,\\
\dot{\theta} &= 2\,d_{12}(\underline{x},t)\;.
\end{aligned}$$



The (Eulerian) strain rate and the Lagrangian strain rate are related by the transport equation

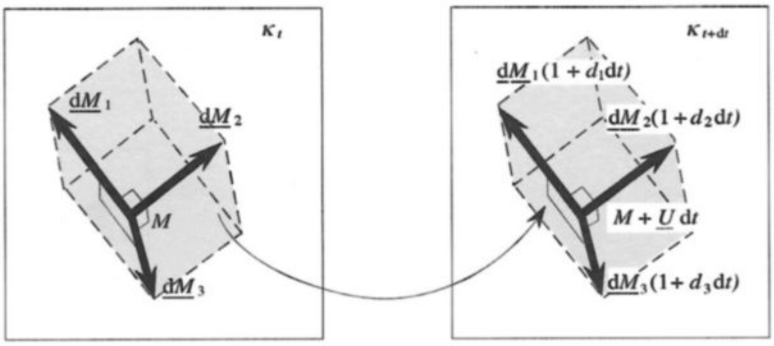
$$\underline{\underline{d}}(\underline{x}, t) = {}^t\underline{\underline{F}}^{-1}(\underline{X}, t) \cdot \dot{\underline{\underline{e}}}(\underline{X}, t) \cdot \underline{\underline{F}}^{-1}(\underline{X}, t)\ ,\ \underline{x} = \underline{\phi}(\underline{X}, t)\ .$$

• In the orthonormal bases of its principal axes in κ_t, $\underline{\underline{d}}(\underline{x}, t)$ takes the diagonal form

$$\underline{\underline{d}}(\underline{x}, t) = d_1(\underline{x}, t)\,\underline{e}_1 \otimes \underline{e}_1 + d_2(\underline{x}, t)\,\underline{e}_2 \otimes \underline{e}_2 + d_3(\underline{x}, t)\,\underline{e}_3 \otimes \underline{e}_3$$

($d_i(\underline{x}, t)$, $i = 1, 2, 3$, principal strain rates).

A triad of material vectors lying along these principal axes in κ_t remains orthogonal in the infinitesimal transformation between t and $t + \mathrm{d}t$ (characterising property).



• The antisymmetric part of the gradient of the velocity field is the *spin tensor*,

$$\underline{\underline{\Omega}}(\underline{x}, t) = \frac{1}{2}\big(\underline{\underline{\mathrm{grad}\,U}}(\underline{x}, t) - {}^t\underline{\underline{\mathrm{grad}\,U}}(\underline{x}, t)\big)\ .$$

It defines locally the angular velocity of the material triad of principal axes of $\underline{\underline{d}}(\underline{x}, t)$, which is also the mean angular velocity of the matter. $\underline{\Omega}(\underline{x}, t)$ is the *spin vector*,

$$\forall\, \underline{\mathrm{d}M} \in \kappa_t\ ,\quad \underline{\underline{\Omega}}(\underline{x}, t) \,.\, \underline{\mathrm{d}M} = \underline{\Omega}(\underline{x}, t) \wedge \underline{\mathrm{d}M}\ .$$

• The Eulerian description, defining physical quantities relative to the current configuration as a function of the space and time variables, does not identify the material elements. The *material derivative* must therefore be carried out as the *derivative with respect to time following the particle or material element in question.*

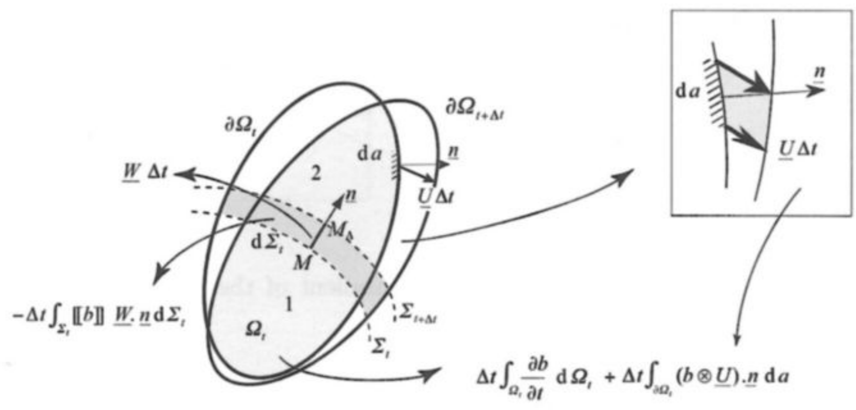
This exhibits the structure of the related equations, where we systematically find a term corresponding to the partial derivative with respect to time, the spatial variables being held constant (the point or geometrical element being frozen, as it were), together with a *convective term.* The latter is the contribution from the convective transport of the particle or material element with which the relevant quantity is associated:

$$\mathcal{B} = b(\underline{x}, t)\ , \qquad \dot{\mathcal{B}} = \frac{\mathrm{d}b}{\mathrm{d}t} = \frac{\partial b}{\partial t} + \operatorname{grad} b \,.\, \underline{U}\ .$$

• Special attention is paid to the material derivative of volume integrals, since these have a privileged role to play in defining physical quantities when a system is modelled by the continuum. We find that

$$\mathcal{I} = \int_{\Omega_t} b(\underline{x}, t)\, \mathrm{d}\Omega_t\ , \quad b \text{ and/or } \underline{U} \text{ discontinuous on } \Sigma_t\ ,$$

$$\dot{\mathcal{I}} = \int_{\Omega_t} \frac{\partial b}{\partial t}\, \mathrm{d}\Omega_t - \int_{\Sigma_t} [\![\, b\,]\!] W\, \mathrm{d}\Sigma_t + \int_{\partial\Omega_t} (b \otimes \underline{U}) \,.\, \underline{\mathrm{d}a}\ .$$



• The mass of a system is given by the integral of its mass per unit volume. The law of *conservation of mass* is expressed by saying that the material derivative of this integral is zero. Use of this global equation leads on a local level to the *continuity equation*, which can be written as a differential equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \,\mathrm{div}\, \underline{U} = 0 \,,$$

and the associated discontinuity equation

$$[\![\rho(\underline{U} - \underline{W})]\!] \,.\, \underline{n} = 0 \quad \text{on } \Sigma_t \,,$$

in the Eulerian description.

The mass of the material element defined by $\mathrm{d}\Omega_0$ at M_0 in the reference configuration κ_0 is

$$\mathrm{d}m = \rho_0(\underline{X})\,\mathrm{d}\Omega_0 = \rho(\underline{x}, t)\,\mathrm{d}\Omega_t \,.$$

In order to model forces, we must begin with a geometrical model of the system and the force model must be consistent with it.

• The example of a system of material points brings out the idea of a *subsystem* within a given system, and the distinction which must be made between external and internal forces acting on the system or subsystem.

The virtual velocities of each of the material points making up the system (or subsystem) generate the *virtual motions* of this system (or subsystem).

For a system or subsystem, the external forces, internal forces and quantities of acceleration (masses times accelerations) define linear forms on the vector space of its virtual motions.

• *The dualised formulation of the fundamental law of dynamics* arises on the vector space of virtual motions. The sum of the virtual rates of work by internal and external forces is equal to the virtual rate of work by quantities of acceleration in the Galilean frame, for any virtual motion of the system (or subsystem):

$$\begin{aligned}
&\text{in a Galilean frame } \mathcal{R}\,,\\
&\forall \mathcal{S}' \subset \mathcal{S}\,,\\
&\forall \hat{\mathbf{U}} \text{ v.m.}\,,\ \mathcal{P}'_{(\mathrm{e})}(\hat{\mathbf{U}}) + \mathcal{P}'_{(\mathrm{i})}(\hat{\mathbf{U}}) = \mathcal{A}'(\hat{\mathbf{U}})\,.
\end{aligned}$$

• *The dualised formulation of the law of mutual actions* refers to *rigid body* virtual motions. The virtual rate of work by internal forces within the system (or subsystem) is zero in any rigid body motion of the system (or subsystem):

$$\begin{aligned}
&\forall \mathcal{S}' \subset \mathcal{S}\,,\\
&\forall \hat{\mathbf{U}} \text{ r.b.v.m. for } \mathcal{S}'\,,\ \mathcal{P}'_{(\mathrm{i})}(\hat{\mathbf{U}}) = 0\,.
\end{aligned}$$

• These two results are then raised to the status of a *principle*. They form the foundation stone for the *virtual work approach* to modelling forces.

• The key role played by rigid body virtual motions leads us to make a special study of them. We introduce the notion of *distributor*,

$$\{\hat{\mathcal{D}}\} = \{\,O\,,\,\underline{\hat{U}}_0\,,\,\underline{\hat{\omega}}_0\,\} = \{\,O'\,,\,\underline{\hat{U}}_0 + \underline{\hat{\omega}}_0 \wedge \underline{OO'}\,,\,\underline{\hat{\omega}}_0\,\}\,,$$

which defines the rigid body virtual velocity field $\underline{\hat{U}}$ by

$$\underline{\hat{U}}(\underline{x}) = \underline{\hat{U}}_0 + \underline{\hat{\omega}}_0 \wedge \underline{OM}$$

The notion of *wrench* then arises by duality,

$$[\mathcal{F}] = [O\,, \underline{F}_0\,, \underline{C}_0] = [O'\,, \underline{F}_0\,, \underline{C}_0 + \underline{O'O} \wedge \underline{F}_0]\,,$$
$$[\mathcal{F}]\,.\,\{\hat{\mathcal{D}}\} = \underline{F}_0\,.\,\underline{\hat{U}}_0 + \underline{C}_0\,.\,\underline{\hat{\omega}}_0\,.$$

• General results are derived, valid for any mechanically consistent model, which express the *fundamental law of dynamics*

$$\begin{cases} \text{in a Galilean frame } \mathcal{R}\,, \\ \forall\, \mathcal{S}' \subset \mathcal{S}\,, \\ [\mathcal{F}'_{\mathrm{e}}] = [\mathcal{M}a']\,, \end{cases}$$

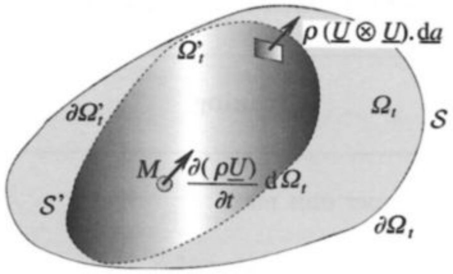
and the *law of mutual actions* in terms of wrenches

$$\forall\, \mathcal{S}' \subset \mathcal{S}\,, \qquad [\mathcal{F}'_{\mathrm{i}}] = 0\,.$$

• If we are concerned in particular with the classical continuum, without further information concerning the model, the fundamental law of dynamics can also be expressed through the *momentum conservation law* in terms of wrenches,

$$\begin{cases} \text{in a Galilean frame } \mathcal{R}\,, \\ \forall\, \mathcal{S}' \subset \mathcal{S}\,, \\ [\mathcal{F}'_{\mathrm{e}}] = \dfrac{\mathrm{d}}{\mathrm{d}t}[\mathcal{M}\mathcal{U}']\,, \end{cases}$$

or by *Euler's theorem*,



in a Galilean frame $\mathcal{R}$,

$$\forall \mathcal{S}' \subset \mathcal{S}, \quad [\mathcal{F}'_{\mathrm{e}}] = \begin{bmatrix} \dfrac{\partial(\rho\underline{U})}{\partial t}\,\mathrm{d}\Omega_t & \text{on } \Omega'_t\,, \\ \rho(\underline{U}\otimes\underline{U})\,.\,\underline{\mathrm{d}a} & \text{on } \partial\Omega'_t\,, \\ (-[\![\rho\,\underline{U}\,]\!]\,W\,\mathrm{d}\Sigma_t \text{ on } \Sigma_t\cap\mathcal{S}' \text{ if shock wave}) & \end{bmatrix}$$

and leads to the *kinetic energy theorem*

$$K'(\underline{U}) = \frac{1}{2}\int_{\Omega'_t} \rho(\underline{x},t)\,\underline{U}^2(\underline{x},t)\,\mathrm{d}\Omega_t\,,$$

in a Galilean frame $\mathcal{R}$,

$$\forall \mathcal{S}' \subset \mathcal{S}\,,\ \mathcal{P}'_{(\mathrm{e})}(\underline{U}) + \mathcal{P}'_{(\mathrm{i})}(\underline{U}) = \frac{\mathrm{d}}{\mathrm{d}t}K'(\underline{U})\,.$$

5. Modelling Forces in Continuum Mechanics

Notation	Meaning
$\underline{F}(\underline{x})$	Body force per unit mass
$\underline{T}_{\Omega}(\underline{x})$	Surface force on boundary of $\mathcal{S}$
$\underline{T}_{\Omega'}(\underline{x})$	Surface force on boundary of $\mathcal{S}'$
$\mathrm{d}\underline{f}$	Infinitesimal surface force
$\underline{\underline{\sigma}}(\underline{x})$	Cauchy stress tensor
$\underline{T}(\underline{x}, \underline{n}(\underline{x}))$	Stress vector
$\underline{\underline{\pi}}(\underline{X}, t)$	Piola–Kirchhoff stress tensor

• In classical continuum mechanics, the force model is built upon a geometrical picture in which the real motions of a system are defined, in the Eulerian description, by the particle velocity field given in the current configuration.

In order to apply the virtual work method, we choose for the vector space of virtual motions the set of continuous and continuously differentiable vector fields (virtual velocities) specified relative to the current configuration of the system. This space is then extended to the *piecewise continuous and continuously differentiable vector fields.*

• External forces on the system are modelled, using the expression for the associated virtual rates of work, by *volume densities of forces* within the system and *surface densities of forces* on its boundary.

The same form is adopted relative to any subsystem, adding the hypothesis that there are *no internal forces at a distance* between the constitutive particles of the system:

$$\mathcal{P}'_{(\mathrm{e})}(\underline{\hat{U}}) = \int_{\Omega'_t} \rho(\underline{x})\underline{F}(\underline{x}) \,.\, \underline{\hat{U}}(\underline{x})\,\mathrm{d}\Omega_t + \int_{\partial\Omega'_t} \underline{T}_{\Omega'}(\underline{x}) \,.\, \underline{\hat{U}}(\underline{x})\,\mathrm{d}a \,.$$

• We postulate that the virtual rate of work by internal forces can be put into the form of a *volume density* which is a linear form in the local values of the virtual velocity field and its gradient, independent of the subsystem under consideration. The principle of virtual work specifies the structure of this density in such a way that the law of mutual actions is satisfied. It allows us to check in an a posteriori manner that the choices made for the various linear forms are mutually compatible.

It shows that the internal forces thus modelled correspond to *contact forces* between the system particles.

• The most general approach in this framework models internal forces by the field of a *second rank tensor*, which must be *symmetric* in order to respect the law of mutual actions. This tensor is dual to the virtual strain rate (up to a sign) in the density of the virtual rate of work by internal forces:

$$\mathcal{P}'_{(\mathrm{i})}(\underline{\hat{U}}) = \int_{\Omega'_t} -\underline{\underline{\sigma}}(\underline{x}) : \underline{\underline{\hat{d}}}(\underline{x})\,\mathrm{d}\Omega_t + \int_{\Sigma_{\underline{\hat{U}}} \cap \Omega'_t} -[\![\underline{\hat{U}}(\underline{x})]\!] \,.\, \underline{\underline{\sigma}}(\underline{x}) \,.\, \underline{n}(\underline{x})\,\mathrm{d}a \,.$$

The equations of motion provide three first order partial differential equations for this symmetric tensor field, viz.,

$$\operatorname{div} \underline{\underline{\sigma}}(\underline{x}) + \rho(\underline{x})\big(\underline{F}(\underline{x}) - \underline{a}(\underline{x})\big) = 0 \quad \text{on } \Omega_t \,,$$

and three boundary conditions on the surface of the system, viz.,

$$\underline{\underline{\sigma}}(\underline{x}) \cdot \underline{n}(\underline{x}) = \underline{T}_{\Omega}(\underline{x}) \quad \text{on } \partial\Omega_t \; .$$

The external forces on the boundary of an arbitrary subsystem are then determined by

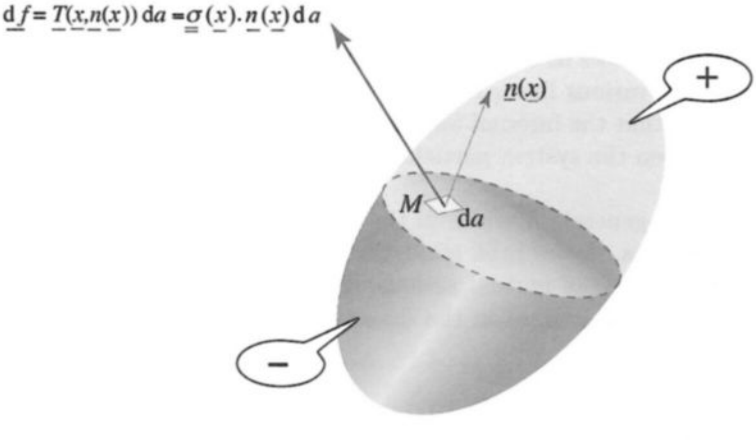
$$\underline{T}_{\Omega'}(\underline{x}) = \underline{\underline{\sigma}}(\underline{x}) \cdot \underline{n}(\underline{x}) \quad \text{on } \partial\Omega'_t \; .$$

• The mechanical interpretation of the model shows that internal forces, reduced to contact forces between the particles of the system, can be viewed schematically in the following way. Particles situated on either side of a geometrical plane within the system, and infinitely close to this plane, exert forces on one another that can be represented by surface forces distributed over the plane,

$$\mathrm{d}\underline{f} = \underline{\underline{\sigma}}(\underline{x}) \cdot \underline{n}(\underline{x})\, \mathrm{d}a \; .$$

Such a plane is called a *facet*. The corresponding surface density is the *stress vector* on the facet,

$$\underline{T}(\underline{x}, \underline{n}(\underline{x})) = \underline{\underline{\sigma}}(\underline{x}) \cdot \underline{n}(\underline{x}) \; .$$

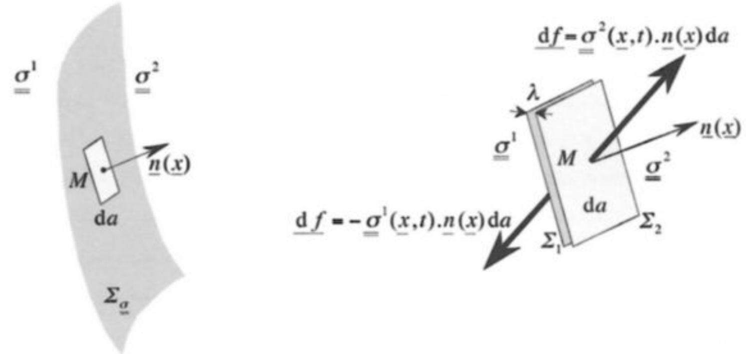


The symmetric tensor field in this model is the *Cauchy stress tensor field*, or (Cauchy) stress field.

• When the stress field is *piecewise* continuous and continuously differentiable the equations of motion also include a discontinuity equation,

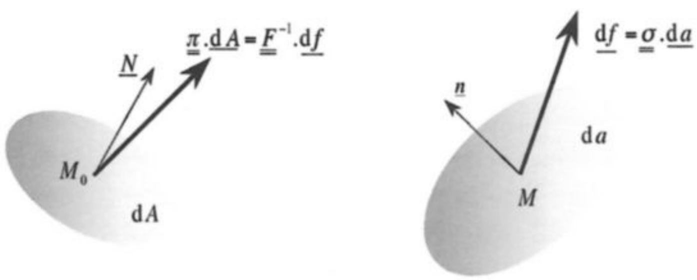
$$[\![\underline{\underline{\sigma}}(\underline{x})]\!] \,.\, \underline{n}(\underline{x}) = 0 \quad \text{on } \Sigma_{\underline{\underline{\sigma}}}$$

provided that there is no shock wave (or surface density of external forces inside the system). This expresses the *continuity of the stress vector* at the discontinuity surface of the field $\underline{\underline{\sigma}}$.



• Considering the integral that gives the virtual rate of work by internal forces, in the absence of velocity discontinuities, and transporting it to the initial reference configuration, the Lagrangian strain rate is found to be associated by duality with a symmetric tensor. From this standpoint, the latter is analogous to the Cauchy stress tensor:

$$\mathcal{P}'_{(\mathrm{i})}(\hat{\underline{U}}) = \int_{\Omega'_0} -\underline{\underline{\pi}}(\underline{X},t) : \hat{\underline{\underline{e}}}(\underline{X},t)\,\mathrm{d}\Omega_0 \;.$$



This is the *Piola–Kirchhoff stress tensor*,

$$\frac{\underline{\underline{\pi}}}{\rho_0} = \underline{\underline{F}}^{-1} \cdot \frac{\underline{\underline{\sigma}}}{\rho} \cdot {}^t\underline{\underline{F}}^{-1} \, ,$$

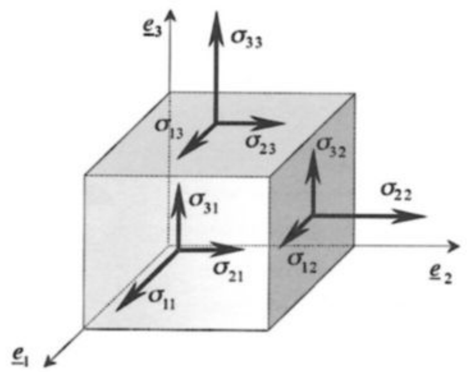
which corresponds to the convective transport of the force on an oriented surface element, as if this force were a material vector. $\underline{\underline{\pi}}\,(\underline{X}, t)$ is symmetric.

6. Local Analysis of Stresses

Notation	**Meaning**
σ	Normal stress
$\underline{\tau}$	Shear stress
$\sigma_1, \sigma_2, \sigma_3$	Principal stresses
I_1, I_2, I_3	Invariants of $\underline{\underline{\sigma}}$
σ_{m}	Mean normal stress
$\underline{\underline{s}}$	Deviator of $\underline{\underline{\sigma}}$
J_2, J_3	Invariants of $\underline{\underline{s}}$
σ_{eq}	Equivalent stress (von Mises)

• The Cauchy stress tensor at a point defines the linear map which determines the stress vector for any facet passing through this point,

$$\underline{T}(\underline{n}) = \underline{\underline{\sigma}} \, . \, \underline{n} \, .$$



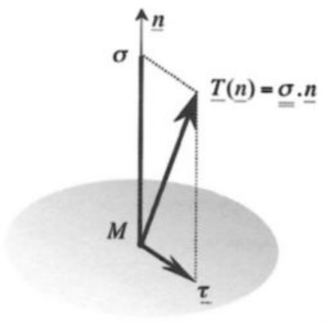
In an orthonormal basis,

$$T_i = \sigma_{ij} \, n_j \ ,$$

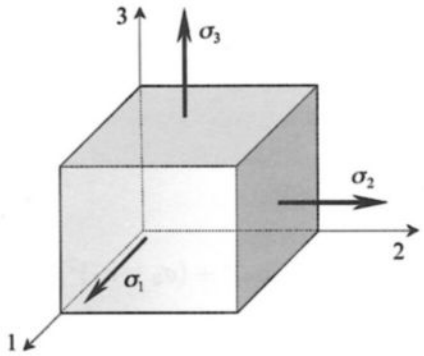
and the following interpretation can be given for the components of $\underline{\underline{\sigma}}$: *the component σ_{ij} of the Cauchy stress tensor represents the component in the direction $\underline{e}_i$ of the stress vector on the facet with normal $\underline{e}_j$.*

• On an arbitrary facet, we calculate the normal stress σ (positive in traction) and the shear stress $\underline{\tau}$:

$$\sigma = \underline{T}(\underline{n}) \, . \, \underline{n} = \underline{n} \, . \, \underline{\underline{\sigma}} \, . \, \underline{n} \, ,$$
$$\underline{T}(\underline{n}) = \sigma \, \underline{n} + \underline{\tau} \, ,$$
$$| \, \underline{\tau} \, | = \left((\underline{\underline{\sigma}} \, . \, \underline{n})^2 - (\underline{n} \, . \, \underline{\underline{\sigma}} \, . \, \underline{n})^2 \right)^{1/2} \, .$$



• In the general case, for a given stress state, there exist three facets on which the stress vector is normal. These facets are orthogonal to the *principal axes* of the stress tensor and the corresponding stresses are called the *principal stresses*.



In the orthonormal basis of principal axes,

$$\underline{\underline{\sigma}} = \sigma_1 \, \underline{e}_1 \otimes \underline{e}_1 + \sigma_2 \, \underline{e}_2 \otimes \underline{e}_2 + \sigma_3 \, \underline{e}_3 \otimes \underline{e}_3 \, .$$

Invariants (under any change of basis) are

$$\begin{aligned} I_1 &= \operatorname{tr} \underline{\underline{\sigma}} &&= \sigma^i{}_i &&= \sigma_1 + \sigma_2 + \sigma_3 \, , \\ I_2 &= \operatorname{tr}(\underline{\underline{\sigma}}^2)/2 &&= (\sigma^i{}_j \, \sigma^j{}_i)/2 &&= \big((\sigma_1)^2 + (\sigma_2)^2 + (\sigma_3)^2\big)/2 \, , \\ I_3 &= \operatorname{tr}(\underline{\underline{\sigma}}^3)/3 &&= (\sigma^i{}_j \, \sigma^i{}_k \, \sigma^k{}_i)/3 &&= \big((\sigma_1)^3 + (\sigma_2)^3 + (\sigma_3)^3\big)/3 \, . \end{aligned}$$

• $\underline{\underline{\sigma}}$ is decomposed into its spherical part $\sigma_{\mathrm{m}} \, \underline{\underline{\mathbb{1}}}$, and its deviatoric part $\underline{\underline{s}}$, or *stress deviator*:

$$\underline{\underline{\sigma}} = \sigma_{\mathrm{m}} \, \underline{\underline{\mathbb{1}}} + \underline{\underline{s}} \, ,$$

$$\sigma_{\mathrm{m}} = I_1/3 \, , \quad \underline{\underline{s}} = \underline{\underline{\sigma}} - \sigma_{\mathrm{m}} \, \underline{\underline{\mathbb{1}}} \, ,$$

with invariants

$$J_1 = \operatorname{tr} \underline{\underline{s}} = 0 \, , \; J_2 = (\operatorname{tr} \underline{\underline{s}}^2)/2 \, , \; J_3 = (\operatorname{tr} \underline{\underline{s}}^3)/3 \, ,$$

and we have the relations

$$\begin{aligned} I_2 &= J_2 + (I_1)^2/6 \, , \\ I_3 &= J_3 + 2 \, I_1 J_2/3 + (I_1)^3/27 \, . \end{aligned}$$

• With regard to the behaviour of the constitutive materials, the stress tensor arises as the load on material elements. The *yield function* is a scalar measure of this load, providing bounds which mark out the domain in which a model based on elastic behaviour is relevant, before irreversible plastic deformations occur. In other words, it determines the *elastic domain* of the material and defines its *yield criterion*.

Two criteria are commonly used for isotropic materials, the Tresca yield criterion,

$$f(\underline{\underline{\sigma}}) = \sup\left\{\sigma_i - \sigma_j - \sigma_0 \,|\, i,j = 1,2,3\right\},$$

and, most often, the von Mises yield criterion

$$f(\underline{\underline{\sigma}}) = \sqrt{J_2} - k = \left\{\frac{1}{6}[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2]\right\}^{\frac{1}{2}} - k,$$

from which we may define the (von Mises) equivalent stress

$$\sigma_{\mathrm{eq}} = \sqrt{3\,J_2}\,.$$

This expresses the magnitude of the simple tension state which is equivalent from the standpoint of the von Mises criteria to the applied stress tensor $\underline{\underline{\sigma}}$.

7. Thermoelasticity

Notation	Meaning
$r(\underline{x}, t)$	Internal heat source per unit volume
$e(\underline{x}, t)$	Internal energy per unit mass
$\underline{q}(\underline{x}, t)$	Outward heat flux
$s(\underline{x}, t)$	Entropy per unit mass
$T(\underline{x}, t)$	Absolute temperature
ψ	Free energy per unit mass
$\varphi_p(\underline{\underline{e}})$	$= 0$, internal constraint
η_p	Lagrange multiplier for an internal constraint
I'_1, I'_2, I'_3	Invariants of $\underline{\underline{e}}$
$\underline{\underline{\underline{\underline{A}}}}$	Elasticity tensor
$\underline{\underline{k}}$	Tensor of thermal coefficients
τ	Temperature variation
λ	Lamé constant
μ, G	Shear modulus
E	Young modulus
ν	Poisson ratio
α	Linear thermal expansion coefficient
K	Bulk modulus

• Experience shows that certain materials have thermoelastic behaviour, characterised by the *reversibility* of their response to conditions they are subject to.

The corresponding model is built upon general principles that apply to any constitutive law, amongst which are the laws of continuum thermodynamics expressed locally by:

the *energy equation* from the first law,

$$\rho\,\dot{e} = \underline{\underline{\sigma}} : \underline{\underline{d}} + r - \operatorname{div} \underline{q}\ ,$$

the *fundamental inequality* from the second law,

$$\rho\,\dot{s} + \operatorname{div}\left(\frac{\underline{q}}{T}\right) - \frac{r}{T} \geq 0\ ,$$

or the *Clausius–Duhem inequality*, obtained by combining with the first,

$$\underline{\underline{\sigma}} : \underline{\underline{d}} - \rho(\dot{\psi} + s\,\dot{T}) - \frac{\underline{q}}{T} \cdot \underline{\operatorname{grad}}\, T \geq 0\ .$$

• The model for thermoelastic behaviour is obtained from the hypothesis that the current values of the temperature and strain tensor of the material element suffice to define its state. The Clausius–Duhem inequality then determines the constitutive law. In the Lagrangian formulation, the free energy is a function of the current values of the temperature and the strain tensor, and arises naturally as the *thermodynamic potential*. The stress tensor is found from it by differentiating with respect to the strain tensor,

$$\underline{\underline{\pi}} = \rho_0 \frac{\partial \psi(T, \underline{\underline{e}})}{\partial \underline{\underline{e}}} \quad \text{for a material without internal constraints}\ .$$

When there are internal constraints, $\varphi_p(\underline{\underline{e}}) = 0$, the constitutive law introduces an arbitrary scalar η_p for each such constraint, so that

$$\begin{cases} \underline{\underline{\pi}} = \rho_0 \dfrac{\partial \psi(T, \underline{\underline{e}})}{\partial \underline{\underline{e}}} + \eta_p \dfrac{\partial \varphi_p(\underline{\underline{e}})}{\partial \underline{\underline{e}}}\ , \\ \varphi_p(\underline{\underline{e}}) = 0\ ,\ p = 1, ..., n\ , \qquad 1 \leq n \leq 6\ . \end{cases}$$

• The constitutive law must respect symmetries of the material.

For an *isotropic* material,

$$\psi(T,\underline{\underline{e}}) = \psi(T, I'_1, I'_2, I'_3)\,,$$
$$I'_1 = \operatorname{tr}\,\underline{\underline{e}}\,,\; I'_2 = \frac{1}{2}\,\operatorname{tr}(\underline{\underline{e}})^2\,,\; I'_3 = \frac{1}{3}\,\operatorname{tr}(\underline{\underline{e}})^3\,.$$

The same is true for internal constraints if these are isotropic,

$$\begin{cases} \underline{\underline{\pi}} = \rho_0 \left(\dfrac{\partial\psi}{\partial I'_1}\,\underline{\underline{1}} + \dfrac{\partial\psi}{\partial I'_2}\,\underline{\underline{e}} + \dfrac{\partial\psi}{\partial I'_3}\,\underline{\underline{e}}^2 \right) + \eta_p \left(\dfrac{\partial\varphi_p}{\partial I'_1}\,\underline{\underline{1}} + \dfrac{\partial\varphi_p}{\partial I'_2}\,\underline{\underline{e}} + \dfrac{\partial\varphi_p}{\partial I'_3}\,\underline{\underline{e}}^2 \right)\,, \\[2ex] \qquad \varphi_p(I'_1\,,\, I'_2\,,\, I'_3) = 0\,,\; p = 1\,,\,\ldots\,,\, n\,,\; 1 \le n \le 3\,. \end{cases}$$

In this case, the tensors $\underline{\underline{\pi}}$ and $\underline{\underline{e}}$ have the same principal axes.

• When *deformations* are *infinitesimal* and the temperature variation *small* (in some well-defined sense), this constitutive law can be linearised. The result is a physical linearisation of the relation between stresses, deformations and temperature variation:

$$\underline{\underline{\pi}} = \underline{\underline{\pi}}^0 + \underline{\underline{\underline{\underline{A}}}} : \underline{\underline{e}} - \underline{\underline{k}}\,\tau\,,$$

where the elasticity tensor $\underline{\underline{\underline{\underline{A}}}}$ has the following symmetries (in an orthonormal basis):

$$A_{ijk\ell} = A_{jik\ell} = A_{ij\ell k} = A_{k\ell ij}\,.$$

• If, moreover, the *transformation is infinitesimal*, linearisation can be taken further. The result is a geometrical linearisation and we arrive at a linear (affine) relation between the Cauchy stress tensor, the linearised strain tensor and the temperature variation from a prestressed reference configuration:

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^{\mathrm{p}} + \underline{\underline{\underline{\underline{A}}}} : \underline{\underline{\varepsilon}}' - \underline{\underline{k}}\,\tau'\,.$$

The linearised constitutive law brings in the *elastic constants* and *thermal expansion coefficients*, intrinsic physical characteristics of the material, whose number is reduced by *material symmetries* and which satisfy the *stability condition* for the material.

• For an *isotropic material*, linear thermoelasticity is characterised by *two* elastic constants, λ (the Lamé constant) and μ (the shear modulus), or by Young's modulus E and the Poisson ratio ν, and also by *a single* thermal expansion coefficient α or the thermal constant k:

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^{\mathrm{p}} + \lambda\,(\mathrm{tr}\,\underline{\underline{\varepsilon}}')\,\underline{\underline{\mathbb{1}}} + 2\mu\,\underline{\underline{\varepsilon}}' - k\,\tau'\,\underline{\underline{\mathbb{1}}}$$

$$\underline{\underline{\varepsilon}}' = \frac{1+\nu}{E}\,\underline{\underline{\sigma}}' - \frac{\nu}{E}\,(\mathrm{tr}\,\underline{\underline{\sigma}}')\,\underline{\underline{\mathbb{1}}} + \alpha\,\tau'\,\underline{\underline{\mathbb{1}}}$$

$$\underline{\underline{\sigma}}' = \underline{\underline{\sigma}} - \underline{\underline{\sigma}}^{\mathrm{p}}$$

$$E = \mu\frac{(3\lambda + 2\mu)}{(\lambda + \mu)} \qquad \nu = \frac{\lambda}{2(\lambda+\mu)}$$

$$\lambda = E\frac{\nu}{(1+\nu)(1-2\nu)} \qquad \mu = \frac{E}{2(1+\nu)}$$

$$k = \frac{E\alpha}{1-2\nu} \qquad \alpha = \frac{k}{3\lambda + 2\mu}$$

$$3\,K = 3\,\lambda + 2\,\mu = \frac{E}{(1-2\nu)}$$

$$3\lambda + 2\mu > 0 \quad \text{and} \quad \mu > 0$$

$$E > 0 \quad \text{and} \quad -1 < \nu < \frac{1}{2}$$

8. Thermoelastic Processes and Equilibrium

Notation	Meaning
$\underline{\underline{K}}^0(T, \underline{\underline{e}})$	Thermal conductivity tensor
T_i^{d}	Specified value of a component of $\underline{T}$
ξ_i^{d}	Specified value of a component of $\underline{\xi}$
S_{T_i}	Portion of boundary where T_i is given
S_{ξ_i}	Portion of boundary where ξ_i is given
$\mathbb{S}(\underline{F}, S_{T_i}, T_i^{\mathrm{d}})$	Set of S.A. stress fields with $\underline{F}$, T_i^{d} on S_{T_i}.
$\mathbb{C}(S_{\xi_i}, \xi_i^{\mathrm{d}})$	Set of K.A. displacement fields with ξ_i^{d} on S_{ξ_i}.
$\mathcal{C}$	Twisting moment
$\varphi(x, y)$	Warping function
α	Differential rotation (angle of twist)
J	Torsional inertia

• Mechanical and thermal excitations are generally imposed on a system as a function of time in the form of volume effects (known body forces) and surface effects, which include the known temperature on the boundary of the system and the *boundary conditions* on the stress vector and the displacement. The *evolution* of a system made up of a thermoelastic material is then specified by a system of equations expressed both in the known initial geometrical configuration of the system and in the current configuration which is an unknown for the evolution problem. These equations are the equations of motion, the continuity equations, the constitutive equation, the heat equation and the boundary conditions.

• When the process is *quasi-static*, this system of equations can be written, at each time,

$$\begin{cases} \operatorname{div} \underline{\underline{\sigma}}(\underline{x},t) + \rho(\underline{x},t)\,\underline{F}(\underline{x},t) = 0 \quad \text{on } \Omega_t \,, \\ [\![\underline{\underline{\sigma}}(\underline{x},t)]\!] \,.\, \underline{n}(\underline{x}) = 0 \quad \text{on } \Sigma_{\underline{\underline{\sigma}}} \,, \end{cases}$$

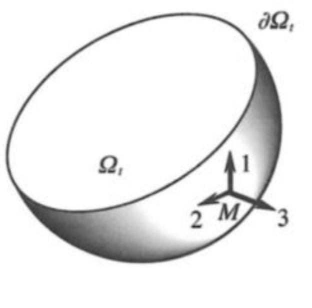
$$\rho_0(\underline{X})/\rho(\underline{x},t) = \det \underline{\underline{F}}(\underline{X},t) \,,$$

$$\begin{cases} \varphi_p(\underline{\underline{e}}) = 0\,,\ p = 1,\dots,n\ ,(1 \leq n \leq 6)\,, \\ \underline{\underline{\pi}} = \rho_0 \dfrac{\partial \psi(T,\underline{\underline{e}})}{\partial \underline{\underline{e}}} + \eta_p \dfrac{\partial \varphi_p(\underline{\underline{e}})}{\partial \underline{\underline{e}}}\,, \quad \eta_p \text{ arbitrary scalars}\,, \end{cases}$$

$$\operatorname{div}_{\underline{X}}\big(\underline{\underline{K}}^0(T,\underline{\underline{e}}) \cdot \underline{\nabla T}\big) = 0 \qquad \text{in the steady regime}\,,$$

$$T(\underline{x},t) = T^{\mathrm{d}}(\underline{x},t) \qquad \text{on } \partial\Omega_t \,,$$

$$\begin{cases} \sigma_{ij}(\underline{x},t)\,n_j(\underline{x}) = T_i^{\mathrm{d}}(\underline{x},t) & \text{on } S_{T_i}(t)\,, \\ U_i(\underline{x},t) = U_i^{\mathrm{d}}(\underline{x},t) & \text{on } S_{U_i}(t)\,, \\ \text{with } S_{U_i}(t) \cup S_{T_i}(t) = \partial\Omega_t \text{ and } S_{U_i}(t) \cap S_{T_i}(t) = \varnothing\ ,\ i = 1,2,3\,. \end{cases}$$



Quasi-static thermoelastic processes can be tackled classically using the displacement method. The uniqueness of the solution is not established.

• A set of hypotheses is grouped together under the heading of the *small perturbation hypothesis* because they enable us to linearise the thermoelastic equilibrium problem. They comprise the small transformation hypothesis, the small displacement hypothesis and the assumption of small temperature variations:

$$\|\underline{\underline{\nabla \xi}}\| \ll 1\,, \qquad \tau \textit{ small}\,, \quad \textit{small displacements}\,.$$

All the equations can then be expressed in the known initial configuration of the system, and they are *linear*.

$$\begin{cases} \operatorname{div} \underline{\underline{\sigma}}(\underline{x},t) + \rho_0(\underline{x})\,\underline{F}(\underline{x},t) = 0 \quad \text{on } \Omega\,, \\ [\![\underline{\underline{\sigma}}(\underline{x},t)]\!]\,.\,\underline{n}(\underline{x}) = 0 \qquad\qquad \text{on } \Sigma_{\underline{\underline{\sigma}}} \end{cases}$$

$$\underline{\underline{\sigma}}(\underline{x},t) = \underline{\underline{\sigma}}^0(\underline{x}) + \underline{\underline{\underline{\underline{A}}}}(\underline{x}) : \underline{\underline{\varepsilon}}(\underline{x},t) - \underline{\underline{k}}(\underline{x})\,\tau(\underline{x},t)$$

$$\operatorname{div}(\underline{\underline{K}}(\underline{x})\,.\,\operatorname{grad} T(\underline{x})) = 0 \quad \text{on } \Omega$$

$$\rho(\underline{x},t) = \rho_0(\underline{x})(1 - \operatorname{tr}\,\underline{\underline{\varepsilon}}(\underline{x},t))$$

$$T(\underline{x},t) = T^{\mathrm{d}}(\underline{x},t) \qquad\qquad \text{on } \partial\Omega\,,$$

$$\begin{cases} \sigma_{ij}(\underline{x},t)\,n_j(\underline{x}) = T_i^{\mathrm{d}}(\underline{x},t) \quad \text{on } S_{T_i} \\ \xi_i(\underline{x},t) = \xi_i^{\mathrm{d}}(\underline{x},t) \qquad\quad \text{on } S_{\xi_i} \\ S_{\xi_i} \cup S_{T_i} = \partial\Omega\,, \quad S_{\xi_i} \cap S_{T_i} = \varnothing\,,\ i = 1,2,3 \end{cases}$$

• The thermal problem *decouples* and can be solved independently. The temperature change field determined in the system at each time becomes a known component of the problem specifications.

At each instant of time, the linearised quasi-static thermoelastic process defines a *thermoelastic equilibrium problem* that only depends on the current excitations and the initial state.

Uniqueness of the solution to this problem is guaranteed when the boundary conditions determine a *well-posed* problem. If the small perturbation hypothesis holds throughout the process, the current state of thermoelastic equilibrium is completely determined when we know the current excitations and the initial state.

The *principle of superposition* expresses linearity of the problem. With the proviso that the small perturbation hypothesis remains valid throughout, the solution to the problem is linearly dependent on the boundary conditions specifying the problem. In practice, the principle of superposition enables us to solve complex problems using linear combinations of solutions to simpler ones.

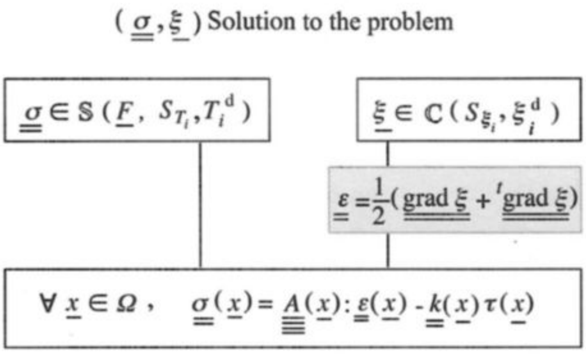
• The solution comprises a *kinematically admissible* displacement field for the problem,

$$\xi_i = \xi_i^{\mathrm{d}} \qquad \text{on } S_{\xi_i} \quad \Leftrightarrow \quad \underline{\xi} \in \mathbb{C}(S_{\xi_i}\,,\, \xi_i^{\mathrm{d}})\,,$$

and a *statically admissible* stress field for the problem,

$$\left.\begin{array}{ll} \operatorname{div} \underline{\underline{\sigma}} + \rho \underline{F} = 0 & (\rho = \rho_0) \\ \sigma_{ij} n_j = T_i^{\mathrm{d}} & \text{on } S_{T_i} \end{array}\right\} \quad \Leftrightarrow \quad \underline{\underline{\sigma}} \in \mathbb{S}(\underline{F}\,,\, S_{T_i}\,,\, T_i^{\mathrm{d}})\,,$$

related by the thermoelastic constitutive law.



• Classical methods for direct solution start out with hypotheses inspired by the form of the boundary conditions and choose one or other of these kinematically or statically admissible fields as the principal unknown. They then express the fact that the field associated with it by the constitutive law is statically or kinematically admissible, respectively.

– The *displacement method* sets out with a postulated expression for $\underline{\xi}$ that is kinematically admissible for the problem. We must then check that the stress field elastically associated with $\underline{\xi}$ is statically admissible for the problem.
  When there is isothermal equilibrium, starting from the natural initial state, in a homogeneous system made from an isotropic material,

we may use the Navier equations

$$(\lambda+\mu)\frac{\partial^2 \xi_j}{\partial x_i \partial x_j} + \mu\, \Delta\xi_i + \rho\, F_i = 0 \qquad i = 1,2,3\,,$$

(in orthonormal Cartesian coordinates) which express the equilibrium equations for the stress field elastically associated with $\underline{\xi}$.

– The *stress method* sets out with a postulated expression for $\underline{\underline{\sigma}}$ that is statically admissible. We must then check that the (linearised) strain field associated with $\underline{\underline{\sigma}}$ is geometrically compatible and generates a kinematically admissible displacement field for the problem.
When there is isothermal equilibrium, starting from the natural initial state, in a homogeneous system made from an isotropic material, we may use the Beltrami–Michell equations

$$(1+\nu)\Delta\sigma_{ij} + \frac{\partial^2(\mathrm{tr}\,\underline{\underline{\sigma}})}{\partial x_i\, \partial x_j} = 0\,, \qquad i = 1,2,3 \quad \text{if} \quad \underline{\underline{\mathrm{grad}\, F}} = 0\,,$$

(in orthonormal Cartesian coordinates) for the geometrical compatibility of the (linearised) strain field elastically associated with $\underline{\underline{\sigma}}$.

– The *uniqueness theorem* justifies a posteriori the hypotheses made at the outset concerning the form chosen for the field, provided they do lead to a solution.

• The linearised elastic equilibrium problem associated with the isothermal *torsion* of a cylindrical rod made from a homogeneous and isotropic material can be solved by either method when the boundary conditions on the end faces assume a well-specified form, related to the geometry of the orthogonal section of the rod.

The stress field solution is a pure shear stress field, not homogeneous over the cross section but invariant under translation along the rod, viz.,

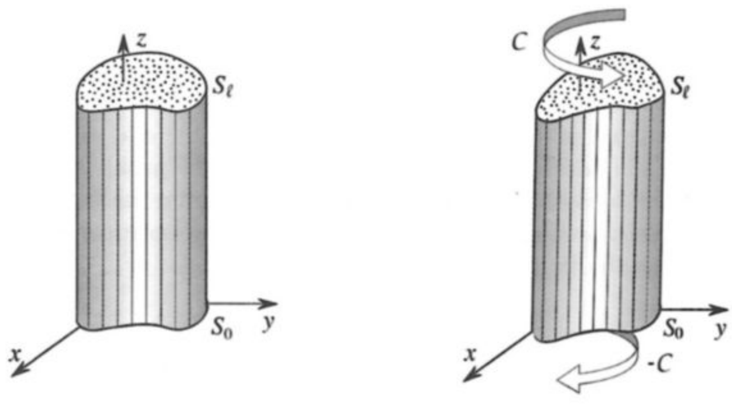
$$\begin{aligned}\underline{\underline{\sigma}} = {} & \mu\,\alpha\left(\frac{\partial\varphi}{\partial x} - y\right)(\underline{e}_x \otimes \underline{e}_z + \underline{e}_z \otimes \underline{e}_x)\\ & + \mu\,\alpha\left(\frac{\partial\varphi}{\partial y} + x\right)(\underline{e}_y \otimes \underline{e}_z + \underline{e}_z \otimes \underline{e}_y)\,.\end{aligned}$$

The displacement field reveals *warping* of the cross sections, as well as differential rotation, both constant along the rod,

$$\underline{\xi} = -\alpha\, z\, y\, \underline{e}_x + \alpha\, z\, x\, \underline{e}_y + \alpha\, \varphi(x,y)\underline{e}_z\,.$$

The *differential rotation* (angle of twist) α is *proportional* to the applied *twisting moment* $\mathcal{C}$:

$$\mathcal{C} = \mu\, J\, \alpha\, .$$



• The *Saint Venant principle* asserts that the solution to the torsion problem obtained for the boundary conditions thus specified on the end sections of the rod remains valid in the *central part* of this same rod, provided it is sufficiently long, for any surface force distribution on the end sections whose *wrench* reduces to a couple with axis parallel to that of the rod. Hence, the precise form of the imposed conditions only introduces effects in the vicinity of the end sections.

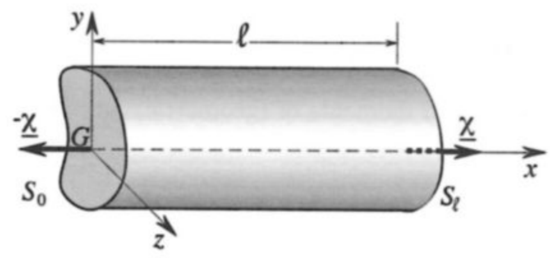
More generally, the Saint Venant principle can be stated for *slender* objects, loaded at the ends.

9. Classic Topics in Three-Dimensional Elasticity

Notation	Meaning
$\underline{\mathcal{X}}$	Resultant on the cross section
$\underline{\mathcal{M}}$	Bending moment
$\mathcal{M}_z$	Component of $\underline{\mathcal{M}}$ along $\underline{e}_z$
$\mathcal{M}_y$	Component of $\underline{\mathcal{M}}$ along $\underline{e}_y$
I_y , I_z	Principal moments of (geometric) inertia for the cross section
$\underline{\omega}(x)$	Rotation of the cross section
χ	Curvature of fibres

Four isothermal elastic equilibrium problems are presented for solids made from a homogeneous and isotropic material. In each case, the initial state of the system under zero loading is taken as reference and assumed natural.

• *Tension–Compression of a Cylindrical Rod*. The solution, obtained for specified boundary conditions at the ends, is extended by the Saint Venant principle to the central part of a slender rod, for which the wrenches of external forces applied at the ends both reduce to a force parallel to the axis of the rod, exerted at the centre of the cross section.



The stress field solution is uniform. It is uniaxial, in tension or compression, and parallel to the rod,

$$\underline{\underline{\sigma}} = E\,\frac{\delta}{\ell}\,\underline{e}_x \otimes \underline{e}_x \,.$$

The displacement field

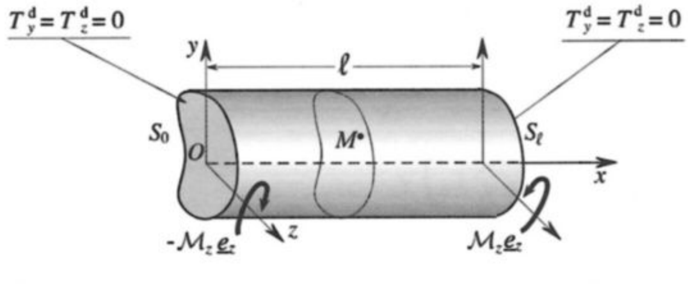
$$\underline{\xi} = \frac{\delta}{\ell}\big(x\,\underline{e}_x - \nu(y\,\underline{e}_y + z\,\underline{e}_z)\big)$$

shows that the rod extends in proportion to its length and the applied tension force,

$$\frac{\mathcal{X}}{S} = E\,\frac{\delta}{\ell}\,.$$

There is also a constant transverse contraction along the rod, proportional to the tension force.

• *Normal Bending of a Cylindrical Rod*. The solution established under particular boundary conditions is extended by the Saint Venant principle to the central part of a sufficiently slender rod when the forces applied to the ends have wrenches that reduce to a moment about a *principal axis of (geometric) inertia* of the cross section.



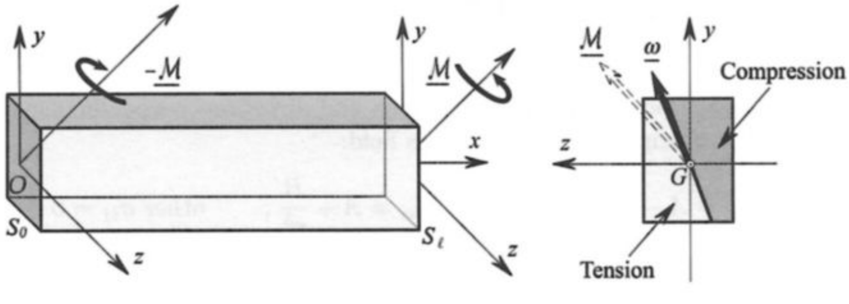
The stress field is uniaxial, in tension or compression, and parallel to the rod. It is invariant under translation along the rod and varies linearly over the cross section, vanishing on the *neutral axis* which coincides with the axis of inertia about which the bending moment is applied,

$$\underline{\underline{\sigma}} = -\frac{\mathcal{M}_z}{I_z}\, y\, \underline{e}_x \otimes \underline{e}_x \, .$$

The displacement field shows that the cross sections remain planar and orthogonal to the mean fibre, which deforms into the arc of a circle. Their rotation about the neutral axis, and also the curvature of the deformed mean fibre, are both proportional to the applied bending moment:

$$\underline{\omega}(x) = \frac{\mathcal{M}_z}{E\, I_z}\, x\, \underline{e}_z \, , \qquad \chi(x) = \frac{\mathrm{d}\omega(x)}{\mathrm{d}x} = \frac{\mathcal{M}_z}{E\, I_z} \, .$$

By linear combination of the solutions obtained for the bending moments about one or other of the principal axes of inertia of the cross section, we can solve the bending problem for a moment of arbitrary direction: $\underline{\mathcal{M}} = \mathcal{M}_y\, \underline{e}_y + \mathcal{M}_z\, \underline{e}_z$.



The solution has the same linearity properties but the neutral axis on which the stress field vanishes and about which the cross section rotates no longer coincides, in general, with the direction of the applied bending moment. The *bending* is said to be *off-axis*:

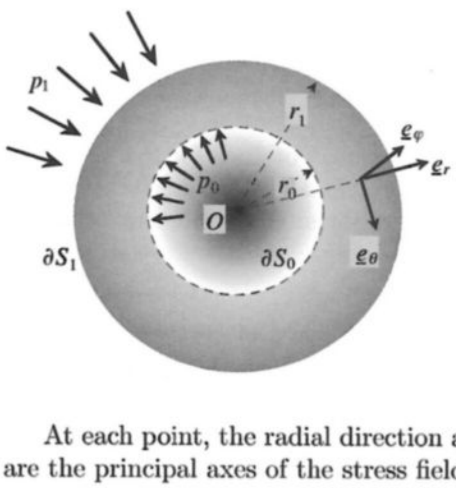
$$\underline{\underline{\sigma}} = \left(-\frac{\mathcal{M}_z}{I_z}\, y + \frac{\mathcal{M}_y}{I_y}\, z \right) \underline{e}_x \otimes \underline{e}_x \,,$$
$$\underline{\omega}(x) = \left(\frac{\mathcal{M}_z}{E\, I_z}\, \underline{e}_z + \frac{\mathcal{M}_y}{E\, I_y}\, \underline{e}_y \right) x \,.$$

• The equilibrium problem of a *spherical shell* subject to *internal and external pressures* is solved by the displacement method.

The solution is spherically symmetric and the displacement field is radial:

$$\underline{\xi} = \left(\frac{A}{3\lambda + 2\mu}\, r + \frac{B}{2\mu\, r^2} \right) \underline{e}_r \,,$$

$$A = \frac{p_0\, r_0^3 - p_1\, r_1^3}{r_1^3 - r_0^3} \,, \qquad B = \frac{1}{2} \frac{(p_0 - p_1)\, r_0^3\, r_1^3}{r_1^3 - r_0^3} \,.$$



At each point, the radial direction and directions perpendicular to it are the principal axes of the stress field:

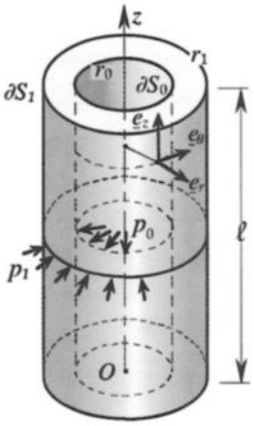
$$\sigma_{rr} = A - 2\frac{B}{r^3} \,, \qquad \sigma_{\theta\theta} = \sigma_{\varphi\varphi} = A + \frac{B}{r^3} \,, \qquad \text{other } \sigma_{ij} = 0 \,.$$

• The equilibrium problem for a *cylindrical tube* under *internal and external pressure* is solved in an analogous way.

The solution is cylindrically symmetric. The displacement field has a radial component that depends on the distance from the tube axis and a component parallel to the tube and proportional to the distance up the axis:

$$\underline{\xi} = \left(\left(\frac{A(1+\nu)(1-2\nu)}{E} - \nu\,\frac{\delta}{\ell} \right) r + \frac{B}{2\mu\, r} \right) \underline{e}_r + z\,\frac{\delta}{\ell}\,\underline{e}_z\,,$$

$$A = \frac{p_0\, r_0^2 - p_1\, r_1^2}{r_1^2 - r_0^2}\,, \qquad B = \frac{(p_0 - p_1) r_0^2\, r_1^2}{r_1^2 - r_0^2}\,.$$



At each point, the directions of the local basis vectors for cylindrical coordinates are the principal axes of the stress field:

$$\sigma_{rr} = A - \frac{B}{r^2}\,, \qquad \sigma_{\theta\theta} = A + \frac{B}{r^2}\,, \qquad \sigma_{zz} = 2\nu\, A + E\,\frac{\delta}{\ell}\,,$$

other $\sigma_{ij} = 0$.

10. Variational Methods in Linearised Thermoelasticity

Notation	Meaning
$\hat{\underline{\xi}}$	Virtual displacement field
$\hat{\underline{\underline{\varepsilon}}}$	Linearised strain associated with $\hat{\underline{\xi}}$
$W(\tau, \underline{\xi}')$	Elastic strain energy of $\underline{\xi}'$
$W(\tau, \underline{\xi}') - \Phi(\underline{\xi}')$	Potential energy of $\underline{\xi}'$
$W^*(\tau, \underline{\underline{\sigma}}')$	Elastic stress energy of $\underline{\underline{\sigma}}'$
$W^*(\tau, \underline{\underline{\sigma}}') - \Phi^*(\underline{\underline{\sigma}}')$	Complementary energy of $\underline{\underline{\sigma}}'$
$\mathbb{A}(S_{T_i})$	Vector space of self-equilibrating stress fields for the problem
$\underline{Q} = (Q_1, \ldots, Q_n)$	Loading parameters
$\underline{q} = (q_1, \ldots, q_n)$	Kinematic parameters
$\mathcal{W}^*(\tau, \underline{Q})$	Elastic stress energy, function of $\underline{Q}$
$\boldsymbol{W}^*(\tau, \underline{Q}', \underline{X}')$	Elastic stress energy, function of $\underline{Q}'$ and $\underline{X}'$

• The *small perturbation hypothesis* allows us to express the principle of virtual work in the initial configuration of the system, and relate a Cauchy stress tensor with a virtual displacement field and its linearised strain field. The corresponding statement is the *virtual work theorem* :

$$\forall \underline{\underline{\sigma}}^* \begin{cases} \text{piecewise continuous and continuously differentiable,} \\ \operatorname{div} \underline{\underline{\sigma}}^* + \rho \underline{F} = 0 \text{ on } \Omega \, , \\ [\![\underline{\underline{\sigma}}^*]\!] \, . \, \underline{n} = 0 \text{ on } \Sigma_{\underline{\underline{\sigma}}^*} \, , \end{cases}$$

$\forall \underline{\hat{\xi}}$ continuous and piecewise continuously differentiable,

$$\begin{aligned} &\int_\Omega \underline{\underline{\sigma}}^*(\underline{x}) : \underline{\underline{\hat{\varepsilon}}}(\underline{x}) \, \mathrm{d}\Omega - \int_\Omega \rho(\underline{x}) \, \underline{F}\,(\underline{x}) \, . \, \underline{\hat{\xi}}\,(\underline{x}) \, \mathrm{d}\Omega \\ &\quad - \int_{\partial\Omega} \underline{\hat{\xi}}\,(\underline{x}) \, . \, \underline{\underline{\sigma}}\,(\underline{x}) \, . \, \underline{n}(\underline{x}) \, \mathrm{d}a = 0 \, . \end{aligned}$$

• This hypothesis also allows us to linearise the thermoelastic constitutive law, which derives from a quadratic potential, a *convex* function of the linearised strains, or its conjugate potential, a *convex* function of the Cauchy stress tensor:

$$\underline{\underline{\sigma}} = \rho \, \frac{\partial \psi(\tau, \underline{\underline{\varepsilon}})}{\partial \underline{\underline{\varepsilon}}} \, ,$$

$$\rho \psi^*(\tau, \underline{\underline{\psi}}) = \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} - \rho \psi(\tau, \underline{\underline{\varepsilon}}) \, ,$$

$$\underline{\underline{\varepsilon}} = \rho \, \frac{\partial \psi(\tau, \underline{\underline{\sigma}})}{\partial \underline{\underline{\sigma}}} \, .$$

• These two arguments are the basis for variational principles. Such principles characterise the displacement and stress field solutions of the linearised thermoelastic equilibrium problem by a *minimum of a convex functional* on the set of kinematically admissible fields or the set of statically admissible fields, respectively. This minimisation guarantees that the stress field or strain field, associated by the constitutive law with the minimising field in the respective cases, satisfies the field equations and boundary conditions relating to it.

• Any displacement field solution minimises the *potential energy* functional over the set of kinematically admissible fields for the boundary conditions specifying the problem:

$$W(\tau, \underline{\xi}') = \int_{\Omega} \rho\psi(\tau, \underline{\underline{\varepsilon}}')\,\mathrm{d}\Omega\,,$$

$$\Phi(\underline{\xi}') = \int_{\Omega} \rho\,\underline{F}\,.\,\underline{\xi}'\,\mathrm{d}\Omega + \sum_i \int_{S_{T_i}} T_i^{\mathrm{d}}\,\xi_i'\,\mathrm{d}a\,,$$

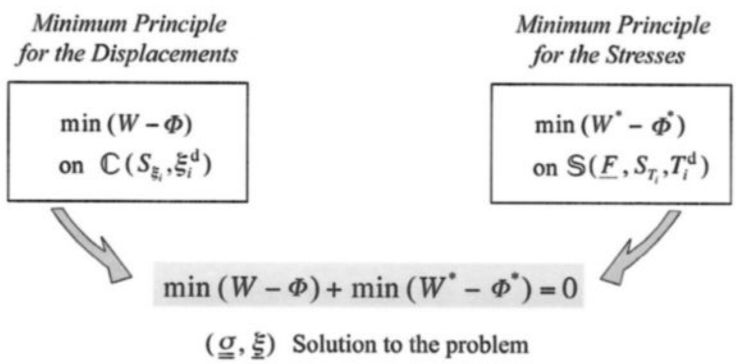
$$(W - \Phi) \quad \text{minimal over} \quad \mathbb{C}(S_{\xi_i}\,,\,\xi_i^{\mathrm{d}})\,.$$

• The stress field solution minimises the *complementary energy* functional over the set of statically admissible fields for the boundary conditions specifying the problem:

$$W^*(\tau, \underline{\underline{\sigma}}') = \int_{\Omega} \rho\psi^*(\tau, \underline{\underline{\sigma}}')\,\mathrm{d}\Omega\,, \quad \Phi^*(\underline{\underline{\sigma}}') = \sum_i \int_{S_{\xi_i}} \xi_i^{\mathrm{d}}\,\sigma_{ij}'\,n_j\,\mathrm{d}a\,,$$

$$(W^* - \Phi^*) \quad \text{minimal over} \quad \mathbb{S}(\underline{F}\,,\,S_{T_i}\,,\,T_i^{\mathrm{d}})\,.$$

• The two variational principles are dual. The minimal values of the two functionals attained for the solution to the problem are equal and opposite.



Simultaneously implementing the variational principles for displacements and stresses leads to energy bounds on the solution $(\underline{\underline{\sigma}}\,,\,\underline{\xi})$:

$$\forall \underline{\xi}' \in \mathbb{C}(S_{\xi_i}\,,\,\xi_i^{\mathrm{d}})\,, \quad \forall \underline{\underline{\sigma}}' \in \mathbb{S}(\underline{F}\,,\,S_{T_i}\,,\,T_i^{\mathrm{d}})\,,$$

$$\begin{aligned} &- W^*(\tau, \underline{\underline{\sigma}}') + \Phi^*(\underline{\underline{\sigma}}') \leq -W^*(\tau, \underline{\underline{\sigma}}) + \Phi^*(\underline{\underline{\sigma}}) \\ &\qquad = W(\tau, \underline{\xi}) - \Phi(\underline{\xi}) \leq W(\tau, \underline{\xi}') - \Phi(\underline{\xi}')\,. \end{aligned}$$

• Apart from demonstrating ***uniqueness*** theorems for the displacement field solution (up to a possible rigid body displacement) and the stress field solution to the problem, these results lead to ***variational methods*** of solution. We explore one or other of the sets of kinematically admissible displacement fields or statically admissible stress fields and minimise the appropriate functional.

Variational methods allow us to introduce the idea of an ***approximate solution.*** They are exploited analytically or *numerically*, particularly by discretising the problem, as in the finite element method.

• In the special case of *isothermal* equilibrium from the *natural initial state*, the values of the elastic strain energy and the elastic stress energy of the system in its present equilibrium state are equal and, by the *Clapeyron equation*, equal to half the work done by all the external forces in the solution displacement field.

Minimum Principle for the Displacements

$$\boxed{\begin{array}{c}\min\,(W-\Phi)\\ \text{on } \mathbb{C}(S_{\xi_i},\xi_i^{\mathrm{d}})\end{array}}$$

Minimum Principle for the Stresses

$$\boxed{\begin{array}{c}\min\,(W^*-\Phi^*)\\ \text{on } \mathbb{S}(\underline{F},S_{T_i},T_i^{\mathrm{d}})\end{array}}$$

$$\min\,(W-\Phi)+\min\,(W^*-\Phi^*)=0$$

$(\underline{\underline{\sigma}},\underline{\xi})$ Solution to the problem

***NATURAL** initial state, **ISOTHERMAL** equilibrium*

$$W(\underline{\xi}) = W^*(\underline{\underline{\sigma}}) = \frac{1}{2}\left(\int_\Omega \rho\,\underline{F}\,.\,\underline{\xi}\,\mathrm{d}\Omega + \int_{\partial\Omega}\underline{T}(\underline{n})\,.\,\underline{\xi}\,\mathrm{d}a\right)$$

The elastic potential $\rho\,\psi(\underline{\underline{\varepsilon}}')$ and conjugate potential $\rho\,\psi^*(\underline{\underline{\sigma}}')$ are then homogeneous quadratic forms. Hence, for an isotropic material,

$$\begin{aligned}
W(\underline{\xi}') &= \int_\Omega \left(\frac{\lambda}{2}(\mathrm{tr}\,\underline{\underline{\varepsilon}}')^2 + \mu\,\mathrm{tr}(\underline{\underline{\varepsilon}}')^2\right)\mathrm{d}\Omega\\
&= \int_\Omega \frac{E}{2(1+\nu)}\left(\frac{\nu}{1-2\nu}(\mathrm{tr}\,\underline{\underline{\varepsilon}}')^2 + \mathrm{tr}(\underline{\underline{\varepsilon}}')^2\right)\mathrm{d}\Omega\,,
\end{aligned}$$

$$\begin{aligned}
W^*(\underline{\underline{\sigma}}') &= \int_\Omega \left(\frac{1+\nu}{2E}\,\mathrm{tr}(\underline{\underline{\sigma}}')^2 - \frac{\nu}{2E}(\mathrm{tr}\,\underline{\underline{\sigma}}')^2\right)\mathrm{d}\Omega\\
&= \int_\Omega \frac{1}{4\mu}\left(\mathrm{tr}(\underline{\underline{\sigma}}')^2 - \frac{\lambda}{(3\lambda+2\mu)}(\mathrm{tr}\,\underline{\underline{\sigma}}')^2\right)\mathrm{d}\Omega\,.
\end{aligned}$$

• Within the same hypothetical framework as for the Clapeyron equation, the Maxwell–Betti *reciprocity theorem* expresses the symmetry existing between two distinct equilibrium states for the same system:

$$\int_{\Omega} \rho \underline{F}^1 \cdot \underline{\xi}^2 \, \mathrm{d}\Omega + \int_{\partial\Omega} \underline{T}^1(\underline{n}) \cdot \underline{\xi}^2 \, \mathrm{d}a = \int_{\Omega} \rho \underline{F}^2 \cdot \underline{\xi}^1 \, \mathrm{d}\Omega + \int_{\partial\Omega} \underline{T}^2(\underline{n}) \cdot \underline{\xi}^1 \, \mathrm{d}a \, .$$

• The set of statically admissible stress fields for the problem is an affine space for which the space of *self-equilibrating stress fields* for the problem is the associated vector space:

$$\mathbb{A}(S_{T_i}) = \mathbb{S}(0\,,\, S_{T_i}\,,\, 0)\ .$$

The self-equilibrating stress fields are statically admissible for the problem with *zero static boundary conditions*. The dimension of this vector space defines the degree of static indeterminacy for the problem.

Expanding the self-equilibrating fields in a basis allows us to define a system of *redundant unknowns*.

• Making the small perturbation hypothesis, we may consider the family of equilibrium problems that depend linearly on the parameters Q_j, $(j = 1, \dots, m)$, for the static boundary conditions, and on q_j, $(j = m+1, \dots, n)$, for the complementary kinematic boundary conditions, for the same system. We define the sets of statically admissible stress fields and kinematically admissible displacement fields $\mathbb{S}$ and $\mathbb{C}$, respectively, for the problem parametrised in this way. The theorem of virtual work then states that

$$\forall \underline{\underline{\sigma}}^* \in \mathbb{S}\,,\ \forall \underline{\hat{\xi}} \in \mathbb{C}\,,\ \int_{\Omega} \underline{\underline{\sigma}}^* : \underline{\underline{\hat{\varepsilon}}} \, \mathrm{d}\Omega = \sum_{j=1}^{n} Q_j^* \, \hat{q}_j = \underline{Q}^* \cdot \underline{\hat{q}}\,,$$

thereby defining complementary parameters Q_j^*, $(m+1, \dots, n)$ and $\hat{q}_j$, $(1, \dots, m)$. The correspondences

$$\underline{\underline{\sigma}}^* \in \mathbb{S} \longrightarrow \underline{Q}^* = \underline{Q}\,(\underline{\underline{\sigma}}^*) \in \mathbb{R}^n\ ,$$

$$\underline{\hat{\xi}} \in \mathbb{C} \longrightarrow \underline{\hat{q}} = \underline{q}\,(\underline{\hat{\xi}}) \in \mathbb{R}^n\ ,$$

are linear. The vectors $\underline{Q}$ and $\underline{q}$ are the *loading parameters* and the *kinematic parameters* for the system, respectively, in the family of problems considered.

• In these circumstances, if the system is made from a thermoelastic material, the *Castigliano theorem* gives an expression for the global thermoelastic constitutive law for the system, relating the parameters $\underline{Q}$ and

$\underline{q}$. It can be written, in an analogous way to the local thermoelastic constitutive law for the material, by means of the global potential $\mathcal{W}^*(\tau, \underline{Q})$, a convex function of $\underline{Q}$, whose value is equal to the elastic stress energy of the system in its thermoelastic equilibrium state under the loading $\underline{Q}$:

$$\underline{q}^{\mathrm{e}}(\underline{Q}) = \frac{\partial \mathcal{W}^*(\tau, \underline{Q})}{\partial \underline{Q}} \,.$$

- The Castigliano theorem assumes that $\mathcal{W}^*(\tau\,,\,\underline{Q}')$ is known, in other words, that the thermoelastic equilibrium problem under the loading $\underline{Q}'$ has been solved. The *minimum potential theorem* allows us to determine this solution when the kinematic boundary conditions are zero and when the vector space of self-equilibrating fields for the parametric problem is finite dimensional. The elastic stress energy of the system is a known convex function of the redundant unknowns $\underline{X}'$ and the loading $\underline{Q}'$: $\boldsymbol{W}^*(\tau, \underline{Q}', \underline{X}')$.

The thermoelastic equilibrium state of the system is obtained from

$$\frac{\partial \boldsymbol{W}^*(\tau, \underline{Q}, \underline{X}^{\mathrm{e}}(\underline{Q}))}{\partial \underline{X}'} = 0 \,,$$

which determines $\underline{X}^{\mathrm{e}}(\underline{Q})$, and then

$$\underline{q}^{\mathrm{e}}(\underline{Q}) = \frac{\partial \boldsymbol{W}^*(\tau, \underline{Q}, \underline{X}^{\mathrm{e}}(\underline{Q}))}{\partial \underline{Q}'} \,.$$

We have

$$\mathcal{W}^*(\tau, \underline{Q}) = \boldsymbol{W}^*\left(\tau, \underline{Q}, \underline{X}^{\mathrm{e}}(\underline{Q})\right) .$$

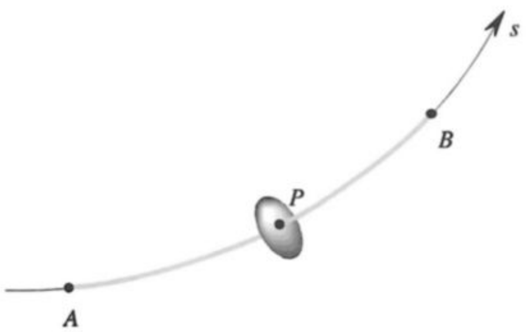
The redundant unknowns can be interpreted as extra loading parameters for which the associated kinematic parameters must be zero in the thermoelastic equilibrium state.

11. Statics of One-Dimensional Media

Notation	Meaning
AB	Arc of director curve: geometrical description of $\mathcal{S}$
$A'B'$	Geometrical description of $\mathcal{S}'$
s	Curvilinear coordinate of field point P along the director curve
$\underline{t}(s)$	Unit vector tangent to the director curve
$\{\hat{\mathbb{U}}\}$	Field of distributors of a virtual motion on AB
$\hat{\underline{U}}(s)\,,\,\hat{\underline{\Omega}}(s)$	Reduced elements of the distributor $\{\hat{\mathbb{U}}(s)\}$ at P
$\hat{\mathbb{U}}$	Virtual motion
$[\mathbf{f}(s)]$	Line density of external force wrenches
$[\mathbb{R}^{\mathcal{S}'}_{A'}]\,,\,[\mathbb{R}^{\mathcal{S}'}_{B'}]$	Wrenches of external forces applied at the endpoints of $\mathcal{S}'$
$\{\hat{\mathbb{D}}(s)\}$	Virtual strain rate of the one-dimensional medium at P
$[\mathbb{X}(s)]$	Internal force wrench
$\underline{\mathcal{X}}(s)\,,\,\underline{\Gamma}(s)$	Reduced elements of $[\mathbb{X}(s)]$ at P
$[\mathbb{F}_i]$	Wrench of concentrated external forces
$N(s)$	Normal force at P
$\underline{V}(s)$	Shearing force at P
$\mathcal{C}(s)$	Twisting moment at P
$\underline{\mathcal{M}}(s)$	Bending moment at P

• Many structures used in industrial practice are made up of slender elements. This geometrical feature suggests implementing a one-dimensional model associated with a *director curve* for analysing such elements.

• In order to account for stiffness in slender three-dimensional elements, the system is modelled as a set of particles specified by their geometrical position on the director curve and by the *orientation* of a *transverse microstructure* associated with each such position. The real motion is described by the field of distributors given at each point by the rigid body motion of the oriented particle specified in this way.



The virtual motions $\hat{\mathbb{U}}$ are defined in a similar way by a piecewise continuous and continuously differentiable *distributor field* along the director curve in the current configuration,

$$\{\,\hat{\mathbb{U}}(s)\,\} = \{\,P\,,\,\underline{\hat{U}}(s)\,,\,\underline{\hat{\Omega}}(s)\,\}\,.$$

• External forces are modelled by the wrenches $[\,\mathbb{R}^{\mathcal{S}'}_{A'}\,]$ and $[\,\mathbb{R}^{\mathcal{S}'}_{B'}\,]$ applied at the endpoints of the system or subsystem, $[\,\mathbb{F}_i\,]$ applied at a discrete set of points in the system, and by a line density of wrenches $[\,\mathsf{f}(s)\,]$:

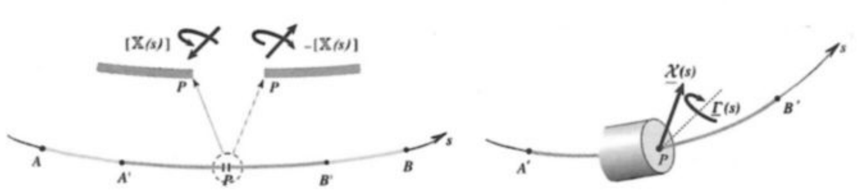
$$\mathcal{P}'_{(\mathrm{e})}(\hat{\mathbb{U}}) = \int_{A'B'} [\,\mathsf{f}(s)\,]\,.\,\{\,\hat{\mathbb{U}}(s)\,\}\,\mathrm{d}s + [\,\mathbb{R}^{\mathcal{S}'}_{A'}\,]\,.\,\{\,\hat{\mathbb{U}}_{A'}\,\} + [\,\mathbb{R}^{\mathcal{S}'}_{B'}\,]\,.\,\{\,\hat{\mathbb{U}}_{B'}\,\}$$

$$+ \sum_{s_{A'}<s_i<s_{B'}} [\,\mathbb{F}_i\,]\,.\,\{\,\hat{\mathbb{U}}(s_i)\,\}\,.$$

- Internal forces are represented by a *field of wrenches*

$$[\mathbb{X}(s)] = [P, \underline{\mathcal{X}}(s), \underline{\Gamma}(s)],$$

which express contact forces between the particles:



$$\mathcal{P}'_{(\mathrm{i})} = -\int_{A'B'} [\mathbb{X}(s)] . \{\hat{\mathbb{D}}(s)\} \, \mathrm{d}s - \sum_{s_{A'} < s_j < s_{B'}} [\mathbb{X}(s_j)] . [\![\{\hat{U}(s_j)\}]\!],$$

$$\{\hat{\mathbb{D}}(s)\} = \frac{\mathrm{d}\{\hat{\mathbb{U}}(s)\}}{\mathrm{d}s}.$$

- The model built in this way is useful for the general study of *beam statics*. It includes the special case of the statics of wires and cables without stiffness.

The equilibrium equations for beams are given, for the field of wrenches $[\mathbb{X}]$, by a differential equation along the director curve, with the associated discontinuity equation and two boundary equations,

$$\frac{\mathrm{d}[\mathbb{X}(s)]}{\mathrm{d}s} + [\mathbf{f}(s)] = 0,$$

$$[\![[\mathbb{X}(s_i)]]\!] + [\mathbb{F}_i] = 0,$$

$$[\mathbb{X}(s_A)] = -[\mathbb{R}_A^{\mathcal{S}}], \qquad [\mathbb{X}(s_B)] = [\mathbb{R}_B^{\mathcal{S}}].$$

External forces at the endpoints of an arbitrary subsystem are then determined by

$$[\mathbb{R}_{A'}] = -[\mathbb{X}(s_{A'})], \qquad [\mathbb{R}_{B'}] = [\mathbb{X}(s_{B'})].$$

Explicitly, we obtain the vector equations

$$\begin{cases} \dfrac{\mathrm{d}\underline{\mathcal{X}}(s)}{\mathrm{d}s} + \underline{f}(s) = 0 \\[2ex] \dfrac{\mathrm{d}\underline{\Gamma}(s)}{\mathrm{d}s} + \underline{t}(s) \wedge \underline{\mathcal{X}}(s) + \underline{m}(s) = 0 \end{cases} \qquad \begin{cases} [\![\underline{\mathcal{X}}(s_i)]\!] + \underline{F}_i = 0 \\[2ex] [\![\underline{\Gamma}(s_i)]\!] + \underline{C}_i = 0 \end{cases}$$

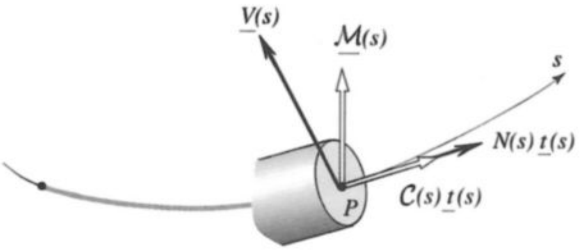
$$\begin{cases} \underline{\mathcal{X}}(s_A) = -\underline{R}_A^{\mathcal{S}} \\[2ex] \underline{\Gamma}(s_A) = -H_A^{\mathcal{S}} \end{cases} \qquad \begin{cases} \underline{\mathcal{X}}(s_B) = \underline{R}_B^{\mathcal{S}} \\[2ex] \underline{\Gamma}(s_B) = \underline{H}_B^{\mathcal{S}} \end{cases}$$

$$\begin{cases} \underline{R}_{A'}^{\mathcal{S}'} = \underline{\mathcal{X}}(s_{A'}) \\[2ex] \underline{H}_{A'}^{\mathcal{S}'} = -\underline{\Gamma}(s_{A'}) \end{cases} \qquad \begin{cases} \underline{R}_{B'}^{\mathcal{S}'} = \underline{\mathcal{X}}(s_{B'}) \\[2ex] \underline{H}_{B'}^{\mathcal{S}'} = \underline{\Gamma}(s_{B'}) \end{cases} .$$

The equilibrium equations for beams can be integrated immediately to give $[\,\mathbb{X}(s)\,]$ *explicitly*:

$$[\,\mathbb{X}(s)\,] = [\,\mathbb{R}_B^{\mathcal{S}}\,] + \int_{PB} [\,\mathsf{f}(\sigma)\,]\,\mathrm{d}\sigma + \sum_{s<s_i<s_B} [\,\mathbb{F}_i\,] \,.$$

• The classic terminology used for the reduced elements of the internal force wrench at the field point on the director curve refers to the transverse microstructure, the *orthogonal cross section*, reflecting the key role it plays in the transmission of forces:



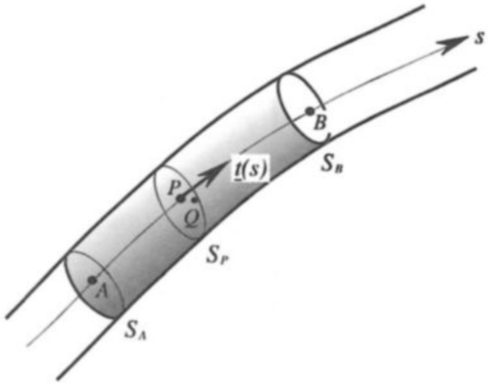
normal force, shearing force,

$$\underline{\mathcal{X}}(s) = N(s)\,\underline{t}(s) + \underline{V}(s) \;;$$

twisting moment, bending moment,

$$\underline{\Gamma}(s) = \mathcal{C}(s)\,\underline{t}(s) + \underline{\mathcal{M}}(s) \,.$$

• Comparing this model with the classical three-dimensional continuum, we can express the internal force field in the one-dimensional medium as a function of the stress field in the slender three-dimensional solid:



$$\begin{aligned}
N(s) &= \int_{S_P} \sigma_{11}(Q)\,\mathrm{d}a\,,\\
\underline{V}(s) &= \int_{S_P} \big(\underline{e}_2\,\sigma_{12}(Q) + \underline{e}_3\,\sigma_{13}(Q)\big)\,\mathrm{d}a\,,\\
\mathcal{C}(s) &= \int_{S_P} \big(x_2\,\sigma_{13}(Q) - x_3\,\sigma_{12}(Q)\big)\,\mathrm{d}a\,,\\
\underline{\mathcal{M}}(s) &= \int_{S_P} (x_3\,\underline{e}_2 - x_2\,\underline{e}_3)\,\sigma_{11}(Q)\,\mathrm{d}a\,.
\end{aligned}$$

• In order to analyse structures, we must define endpoint, support and assembly boundary conditions for the constitutive members. Explicit integration of the static equations shows that the *degree of static indeterminacy* of the structures modelled in this way, when there are no continuous supports, is *always finite*. From the kinematic point of view, geometrical compatibility conditions concern the respect for assembly boundary conditions and support boundary conditions. They are conveniently expressed in dualised form.

12. Thermoelastic Structural Analysis

Notation	Meaning
$\{\boldsymbol{\xi}\}$	Distributor field of real displacements
$\underline{\xi}(s)\,,\ \underline{\omega}(s)$	Reduced elements of $\{\boldsymbol{\xi}(s)\}$ at P
$\omega_t(s)$	Component of $\underline{\omega}(s)$ along $\underline{t}(s)$
$\underline{\omega}_\perp(s)$	Component of $\underline{\omega}(s)$ in the plane normal to $\underline{t}(s)$
$\{\boldsymbol{\mathcal{E}}(s)\}$	Deformation of one-dimensional medium at P
$\varepsilon(s)$	Extension per unit length of the direction curve
$N^{\mathrm{p}}\,,\ \underline{V}^{\mathrm{p}}\,,\ \underline{\mathcal{M}}^{\mathrm{p}}\,,\ \mathcal{C}^{\mathrm{p}}$	Reduced elements for the prestressed initial state
$\tau(s)$	Temperature variation
$w^*\Big([\mathbb{X}(s)], \underline{t}(s), \underline{e}_y(s), \underline{e}_z(s)\Big)$	Line density of elastic stress energy for the one-dimensional medium

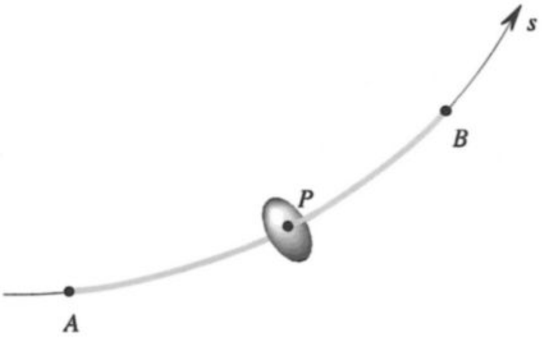
• Having set up a geometrical model and force model for the one-dimensional medium, the analysis of structures also requires a constitutive law.

• The assumption of *infinitesimal transformations* for the one-dimensional medium allows us to introduce displacement and strain distributors,

$$\{\boldsymbol{\xi}(s)\} = \{\, P\,,\, \underline{\xi}(s)\,,\, \underline{\omega}(s)\,\} = \{\, P\,,\, \underline{\xi}(s)\,,\, \omega_t(s)\,\underline{t}(s) + \underline{\omega}_\perp(s)\,\}\,,$$

$$\{\mathcal{E}(s)\} = \frac{\mathrm{d}\{\boldsymbol{\xi}(s)\}}{\mathrm{d}\,s}\,.$$

• The linear thermoelastic constitutive law, which satisfies the general principles stated in the case of the three-dimensional thermoelastic continuum, is a linear relation between the strain distributor, the internal force wrench and the temperature variation at the field point. This relation is oriented in space. It is the components of the reduced elements of the strain distributor and the internal force wrench along the tangent to the direction curve and along the axes specifying the orientation of the transverse microstructure which enter explicitly into the constitutive law.



• The constitutive law commonly used for one-dimensional media in *structural analysis* when modelling rods or, more generally, slender arches with low curvature and slowly varying cross section, assumes that real motions of the one-dimensional medium satisfy the *Navier–Bernoulli condition*. This says that the displacement $\underline{\xi}$ of the direction curve and the rotation $\underline{\omega}$ of the microstructure represented, for example, by the orthogonal section are related by

$$\underline{\omega}_{\perp}(s) = \underline{t}(s) \wedge \frac{\mathrm{d}\underline{\xi}(s)}{\mathrm{d}s} ,$$

in such a way that the orthogonal section remains orthogonal to the direction curve.

The strain distributor can then be written

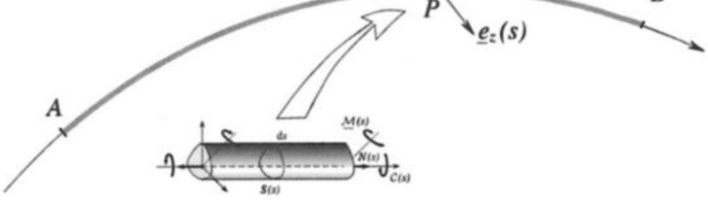
$$\{\mathcal{E}(s)\} = \{\, P \,,\, \varepsilon(s)\,\underline{t}(s) \,,\, \frac{\mathrm{d}\underline{\omega}(s)}{\mathrm{d}s} \,\} ,$$

where

$$\varepsilon(s) = \frac{\mathrm{d}\underline{\xi}(s)}{\mathrm{d}s} \cdot \underline{t}(s) , \qquad \frac{\mathrm{d}\underline{\xi}(s)}{\mathrm{d}s} = \varepsilon(s)\,\underline{t}(s) + \underline{\omega}(s) \wedge \underline{t}(s) .$$

• The constitutive law can be identified for a straight beam element (one-dimensional medium) by appealing to the solution of the *Saint Venant problem* for a homogeneous cylindrical rod, made from an isotropic linear thermoelastic material, subject to a combined bending and a torsion. This determines the position of the direction curve of the beam element. In fact, it must follow the mean fibre of the cylindrical rod. The microstructure of the one-dimensional medium, for its part, is oriented by the principal axes of inertia of the orthogonal section of this same rod.

This constitutive law is then adopted for the one-dimensional model of straight rods and slender arches with *low curvature* and constant or *slowly varying* cross section. Such an arch is thus modelled by a heterogeneous curved beam, whose microstructure is oriented at each point by the *principal axes of inertia of the orthogonal section* of the arch, and which behaves like the homogeneous straight beam modelling a cylindrical rod of the same cross section as the arch at the point in question.



• It is convenient to express this linearised thermoelastic constitutive law in terms of the elastic stress energy of the one-dimensional medium:

$$
\begin{aligned}
w^*\Big([\mathbb{X}(s)]\,,\underline{t}(s),\underline{e}_y(s),\underline{e}_z(s)\Big) &= \frac{\Big(N(s)-N^{\mathrm{p}}(s)\Big)^2}{2\,E\,S(s)} \\
&+ \frac{\Big(\mathcal{M}_y(s)-\mathcal{M}_y^{\mathrm{p}}(s)\Big)^2}{2\,E\,I_y(s)} + \frac{\Big(\mathcal{M}_z(s)-\mathcal{M}_z^{\mathrm{p}}(s)\Big)^2}{2\,E\,I_z(s)} \\
&+ \frac{\Big(\mathcal{C}(s)-\mathcal{C}^{\mathrm{p}}(s)\Big)^2}{2\,\mu\,J(s)} + \Big(N(s)-N^{\mathrm{p}}(s)\Big)\,\alpha\,\tau(s)\,,
\end{aligned}
$$

$$
\{\varepsilon(s)\} = \frac{\partial w^*\Big([\mathbb{X}(s)]\,,\underline{t}(s)\,,\,\underline{e}_y(s)\,,\,\underline{e}_z(s)\Big)}{\partial[\mathbb{X}(s)]}\,,
$$

which becomes, in detail,

$$
\begin{aligned}
\varepsilon(s) &= \frac{N(s)-N^{\mathrm{p}}(s)}{E\,S(s)} + \alpha\,\tau(s)\,, \\
\frac{\mathrm{d}\underline{\omega}(s)}{\mathrm{d}s} &= \frac{\mathcal{M}_y(s)-\mathcal{M}_y^{\mathrm{p}}(s)}{E\,I_y(s)}\,\underline{e}_y(s) + \frac{\mathcal{M}_z(s)-\mathcal{M}_z^{\mathrm{p}}(s)}{E\,I_z(s)}\,\underline{e}_z(s) \\
&+ \frac{\mathcal{C}(s)-\mathcal{C}^{\mathrm{p}}(s)}{\mu\,J(s)}\,\underline{t}(s)\,.
\end{aligned}
$$

• The *small perturbation hypothesis* allows us to linearise the thermoelastic equilibrium problem for structures composed of one-dimensional members. It must be carefully justified by checking that displacements do indeed remain small. Under this hypothesis, the *uniqueness* results established in the context of three-dimensional thermoelastic equilibrium problems still hold. Without this hypothesis, however, we may well have to face *stability* problems, in which different solutions can be found for the same problem.

• *Statically determinate* problems are solved by appealing to the constitutive law for the one-dimensional medium in order to determine the deformations and displacements within the structure. The *Castigliano theorem* is often used.

• For *statically indeterminate* problems, the constitutive law is required to determine internal forces, deformations and displacements. We may proceed directly by expressing compatibility between *deformations* of the members making up the structure, the *continuity* of the structure, and the *support boundary conditions*. We may also use the *minimum potential theorem* which expresses these compatibility conditions in dualised form.

Useful Formulas in Cylindrical Coordinates (Orthonormal Basis $\underline{e}_r$, $\underline{e}_\theta$, $\underline{e}_z$)

Kinematics

$$\underline{U} = U_r \, \underline{e}_r + U_\theta \, \underline{e}_\theta + U_z \, \underline{e}_z$$

$$d_{rr} = \frac{\partial U_r}{\partial r}, \qquad d_{\theta\theta} = \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r}, \qquad d_{zz} = \frac{\partial U_z}{\partial z}$$

$$d_{r\theta} = \frac{1}{2}\left(\frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r} + \frac{1}{r}\frac{\partial U_r}{\partial \theta}\right)$$

$$d_{\theta z} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial U_z}{\partial \theta} + \frac{\partial U_\theta}{\partial z}\right), \qquad d_{zr} = \frac{1}{2}\left(\frac{\partial U_r}{\partial z} + \frac{\partial U_z}{\partial r}\right)$$

$$\operatorname{tr} \underline{\underline{d}} = \operatorname{div} \underline{U} = \frac{\partial U_r}{\partial r} + \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r} + \frac{\partial U_z}{\partial z}$$

Linearised strains (infinitesimal transformations)

$$\underline{\xi} = \xi_r \, \underline{e}_r + \xi_\theta \, \underline{e}_\theta + \xi_z \underline{e}_z$$

$$\varepsilon_{rr} = \frac{\partial \xi_r}{\partial r}, \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial \xi_\theta}{\partial \theta} + \frac{\xi_r}{r}, \qquad \varepsilon_{zz} = \frac{\partial \xi_z}{\partial z}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial \xi_\theta}{\partial r} - \frac{\xi_\theta}{r} + \frac{1}{r}\frac{\partial \xi_r}{\partial \theta}\right)$$

$$\varepsilon_{\theta z} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial \xi_z}{\partial \theta} + \frac{\partial \xi_\theta}{\partial z}\right), \qquad \xi_{zr} = \frac{1}{2}\left(\frac{\partial \xi_r}{\partial z} + \frac{\partial \xi_z}{\partial r}\right)$$

$$\operatorname{tr} \underline{\underline{\varepsilon}} = \operatorname{div} \underline{\xi} = \frac{\partial \xi_r}{\partial r} + \frac{1}{r}\frac{\partial \xi_\theta}{\partial \theta} + \frac{\xi_r}{r} + \frac{\partial \xi_z}{\partial z}$$

Equations of motion

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho(F_r - a_r) = 0$$

$$\frac{\partial \sigma_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + 2\frac{\sigma_{r\theta}}{r} + \rho(F_\theta - a_\theta) = 0$$

$$\frac{\partial \sigma_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{zr}}{r} + \rho(F_z - a_z) = 0$$

Useful Formulas in Spherical Coordinates (Orthonormal Basis $\underline{e}_r$, $\underline{e}_\theta$, $\underline{e}_\varphi$)

Kinematics

$$\underline{U} = U_r\underline{e}_r + U_\theta\underline{e}_\theta + U_\varphi\underline{e}_\varphi$$

$$d_{rr} = \frac{\partial U_r}{\partial r}, \qquad d_{\theta\theta} = \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r}$$

$$d_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial U_\varphi}{\partial \varphi} + \frac{U_\theta}{r}\cot\theta + \frac{U_r}{r}$$

$$d_{r\theta} = \frac{1}{2}\left(\frac{\partial U_r}{r\partial \theta} + \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r}\right)$$

$$d_{\theta\varphi} = \frac{1}{2}\left(\frac{\partial U_\varphi}{r\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial U_\theta}{\partial \varphi} - \frac{\cot\theta}{r}U_\varphi\right)$$

$$d_{\varphi r} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial U_r}{\partial \varphi} + \frac{\partial U_\varphi}{\partial r} - \frac{U_\varphi}{r}\right)$$

$$\mathrm{tr}\,\underline{\underline{d}} = \mathrm{div}\,\underline{U} = \frac{\partial U_r}{\partial r} + \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial U_\varphi}{\partial \varphi} + \frac{U_\theta}{r}\cot\theta + 2\frac{U_r}{r}$$

Linearised strains (infinitesimal transformations)

$$\underline{\xi} = \xi_r\,\underline{e}_r + \xi_\theta\,\underline{e}_\theta + \xi_\varphi\,\underline{e}_\varphi$$

$$\varepsilon_{rr} = \frac{\partial \xi_r}{\partial r}, \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial \xi_\theta}{\partial \theta} + \frac{\xi_r}{r}$$

$$\varepsilon_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial \xi_\varphi}{\partial \varphi} + \frac{\xi_\theta}{r}\cot\theta + \frac{\xi_r}{r}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial \xi_r}{r\partial \theta} + \frac{\partial \xi_\theta}{\partial r} - \frac{\xi_\theta}{r}\right)$$

$$\varepsilon_{\theta\varphi} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial \xi_\varphi}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \xi_\theta}{\partial \varphi} - \frac{\cot\theta}{r}\xi_\varphi\right)$$

$$\varepsilon_{\varphi r} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial \xi_r}{\partial \varphi} + \frac{\partial \xi_\varphi}{\partial r} - \frac{\xi_\varphi}{r}\right)$$

$$\mathrm{tr}\,\underline{\underline{\varepsilon}} = \mathrm{div}\,\underline{\xi} = \frac{\partial \xi_r}{\partial r} + \frac{1}{r}\frac{\partial \xi_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \xi_\varphi}{\partial \varphi} + \frac{\xi_\theta}{r}\cot\theta + 2\frac{\xi_r}{r}$$

Equations of motion

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\,\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{1}{r\sin\theta}\,\frac{\partial\sigma_{r\varphi}}{\partial\varphi}+\frac{1}{r}(2\sigma_{rr}-\sigma_{\theta\theta}-\sigma_{\varphi\varphi}+\sigma_{r\theta}\cot\theta)$$
$$+\rho(F_r-a_r)=0$$

$$\frac{\partial\sigma_{\theta r}}{\partial r}+\frac{1}{r}\,\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{1}{r\sin\theta}\,\frac{\partial\sigma_{\theta\varphi}}{\partial\varphi}+\frac{1}{r}\Big((\sigma_{\theta\theta}-\sigma_{\varphi\varphi})\cot\theta+3\sigma_{r\theta}\Big)$$
$$+\rho(F_\theta-a_\theta)=0$$

$$\frac{\partial\sigma_{\varphi r}}{\partial r}+\frac{1}{r}\,\frac{\partial\sigma_{\varphi\theta}}{\partial\theta}+\frac{1}{r\sin\theta}\,\frac{\partial\sigma_{\varphi\varphi}}{\partial\varphi}+\frac{1}{r}(3\sigma_{\varphi r}+2\sigma_{\varphi\theta}\cot\theta)$$
$$+\rho(F_\varphi-a_\varphi)=0$$

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Mechanics

Cover pictures:

Centre : Solférino footbridge (Paris) showing structural elements. Photo : J. Salençon.

Left : Stress visualisation using photoelasticity. Stress concentrations around a notch on a test piece in tension. Photo : J. Salençon.

Right : Visualising aerodynamic flows using laser tomography in an experimental wind-tunnel at ONERA.

ÉCOLE POLYTECHNIQUE

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