Following Galileo, from "Matter" to "Machines"

Jean Salençon

Abstract

The two first "Days" of Galileo's *Dialogues concerning two new sciences* (1638) are devoted to what is now commonly called the Strength of materials, in an attempt to derive the resistance of simple structures (*"machines"*) from what he defines as the resistance of their constituent materials (*"matter"*). His rationale consists in stating that machine equilibrium and matter resistance must be compatible, an argument that became the cornerstone of stability analyses in civil engineering for centuries, up to the present theory of Yield design and subsequent developments. Within the same framework, Galileo was concerned about size effects on the design of structures: he is considered as having provided the first example of Dimensional analysis, a theory that has been developed since the end of the 19th century and found its full mathematical proof in the mid-20th, with current applications to reduced scale modelling.

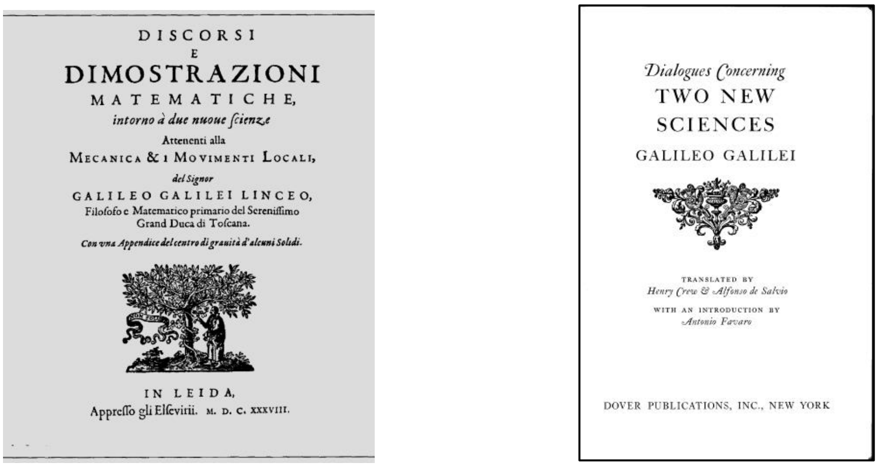
1. Galileo's Dialogues

According to his writings to his faithful friend Elia DIODATI, GALILEO considered the *New Sciences* as "*Superior to everything else of mine hitherto published… They contain results which I consider the most important of all my studies*". The foundations of this work were laid as early as Galileo's 18-year stay in Padua from 1592 when he had been appointed professor of mathematics at the University, but it was given its final form during Galileo's 5-month enforced residence in Siena after his trial and condemnation in 1633. Publication of the work in Italy proved impossible due to the express order prohibiting printing or reprinting any work by GALILEO, either in Venice or in any other place, *nullo excepto*. This is the reason why this celebrated book appeared in Leyden after Louis ELZEVIR had arrived in Italy in 1636 and taken the manuscript with him on his return home[1].

But for the Appendix devoted to "*some theorems and their proofs, dealing with centres of gravity of solid bodies*", the book is organised in *Four Days*. The 1st and 2nd Days are concerned with the first New Science referred to in the title, namely what would now be called *Strength of materials*, while the 3rd and 4th ones deal with the second New Science, *Kinematics and Dynamics*.

[1] This short introduction is derived from the preface written by Professor Antonio FAVARO of the University of Padua to the 1st edition of the English translation [2] published in 1914.

Recalling VOLTAIRE's saying about the secret of boring[2] "*Le secret d'ennuyer est celui de tout dire*", we will just pick up, in that book, a couple of seminal topics that were developed during the following centuries and found practical applications in the design of structures. For this purpose, excerpts will be quoted from the English translation of the *Discorsi e Dimostrazioni Matematiche intorno à due nuove scienze* (Galileo, 1638) by Henry CREW and Alfonso de SALVIO with *Dialogues concerning two New Sciences* as a title (Figure 1)[3].


DISCORSI
E
DIMOSTRAZIONI
MATEMATICHE,
intorno à due nuove scienze
Attenenti alla
MECANICA & I MOVIMENTI LOCALI,
del Signor
GALILEO GALILEI LINCEO,
Filosofo e Matematico primario del Serenissimo
Grand Duca di Toscana.
Con una Appendice del centro di grauità d'alcuni Solidi.
IN LEIDA,
Appresso gli Elsevirii. M. D. C. XXXVIII.

Dialogues Concerning
TWO NEW
SCIENCES
GALILEO GALILEI
TRANSLATED BY
Henry Crew & Alfonso de Salvio
WITH AN INTRODUCTION BY
Antonio Favaro
DOVER PUBLICATIONS, INC., NEW YORK

Figure 1. Galileo's *Discorsi*

2. Galileo's Approach to the Strength of Matter

The *First Day* starts with considerations where the size of a "*machine*" or structure is discussed as depending on the matter it is made from. Schematically the question is about the possibility of extrapolating a reduced scale structure to a real size one by enlarging it through pure geometrical similarity. Through the voice of Salviati, one of the three dialogists, GALILEO denies the feasibility of such a process and delivers his answer:

> *"Yet I shall say and will affirm that, even if the imperfections did not exist and matter were absolutely perfect, unalterable and free from all accidental variations, still the mere fact that it is matter makes the larger machine, built of the same material and in the same proportion as the smaller, correspond with exactness to the smaller in every respect except that it will not be so strong or so resistant against violent treatment; the larger the machine, the greater its weakness."*

[2] (Voltaire, 1737)
[3] References to Figures and pages are given according to this English edition.

where the so-called *"violent treatment"* is implicitly just related to the action of gravity forces in the "machine" or structure. After what, in the same dialogue, Salviati presents convincing arguments among which (p. 52):

> *"I am certain you both know that* […] *nature cannot produce a horse as large as twenty ordinary horses or a giant ten times taller than an ordinary man unless by miracle or by greatly altering the proportions of his limbs and especially of his bones, which would have to be considerably enlarged over the ordinary."*

Without being explained explicitly at that stage, the competition between the action of gravity forces and some kind of resistance forces in the "machine" or system is already underlying here. The statement that *"the larger the machine, the greater its weakness"* leads us to guessing that these two kinds of forces are related in a different way to the geometrical scale of the solid under consideration but it is only in the *Second Day* that the concept of *"resistance* [resistenza] *that solids offer to fracture"* is clearly introduced and thoroughly dealt with, referring to longitudinal pull tests performed on cylindrical samples up to fracture (Figure 2).

The strength so defined is the fracture force related to the considered cylinder (pp. 156-157):

> *"If now we define absolute resistance to fracture as that offered to a longitudinal pull (in which case the stretching force acts in the same direction as that through which the body is moved)".*

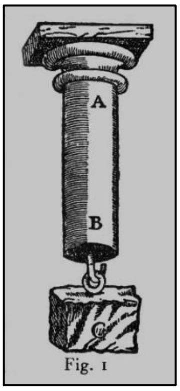


Figure 2. Galileo's *Discorsi*, (p. 55)

Then, somehow anticipating on the concept of stress that would be introduced two centuries later, Salviati (GALILEO) recognises that (Figure 3)

> *"If we consider the resistance to fracture by longitudinal pull as dependent upon bases… no one can doubt that the strength of the cylinder B is greater than that of A*

in the same proportion in which the area of the circle EF exceeds that of CD; because it is precisely in this ratio that the number of fibres binding the parts of the solid together in the one cylinder exceeds that in the other cylinder."

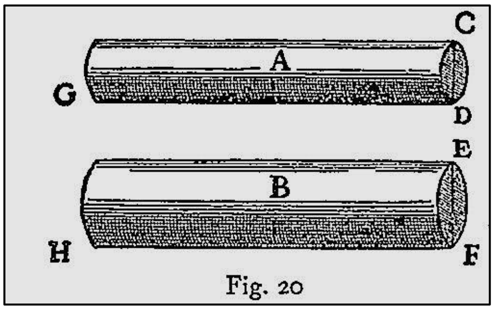


Figure 3. Galileo's *Discorsi*, (p. 160)

This being acknowledged together with the fact that gravity forces obviously refer to volumes the conclusion comes as (p. 169):

"From what has already been demonstrated, you can plainly see the impossibility of increasing the size of structures to vast dimensions either in art or in nature; … ; nor can nature produce trees of extraordinary size because the branches would break down under their own weight; so it would be impossible to build up the bony structures of men, horses, or animals so as to hold together and perform their normal functions if these animals were to be increased enormously in height; for this increase in height can be accomplished only by employing a material which is harder and stronger than usual, or by enlarging the size of the bones, thus changing their shape until the form and appearance of the animals suggest a monstrosity."

"To illustrate briefly, I have sketched a bone whose natural length has been increased three times and whose thickness has been multiplied until, for a correspondingly large animal, it would perform the same function which the small bone performs for its small animal." (Figure 4).

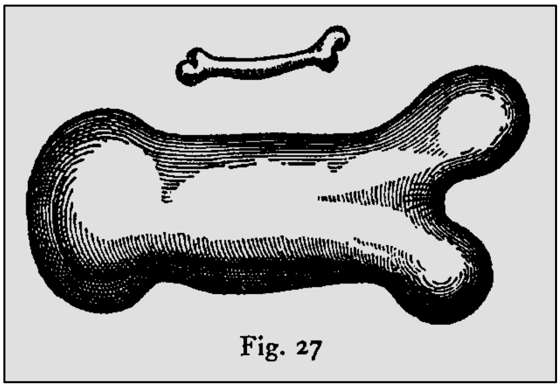


Figure 4. Galileo's *Discorsi*, (p.170)

Upon the same premises, where the strength of a cylindrical rod is defined as the resistance to a longitudinal pull test, GALILEO, through Salviati's voice, endeavours to explain why (p. 156)

> *"A prism or solid cylinder of glass, steel, wood or other breakable material which is capable of sustaining a very heavy weight when applied longitudinally is, as previously remarked, easily broken by a transverse application of a weight which may be much smaller in proportion as the length of the cylinder exceeds its thickness"*

through the analysis of the cantilever beam problem (Figure 5):

> *"Let us imagine a solid prism ABCD fastened into a wall at the end AB, and supporting a weight E at the other end; understand also that the wall is vertical and that the prism or cylinder is fastened at right angles to the wall."*

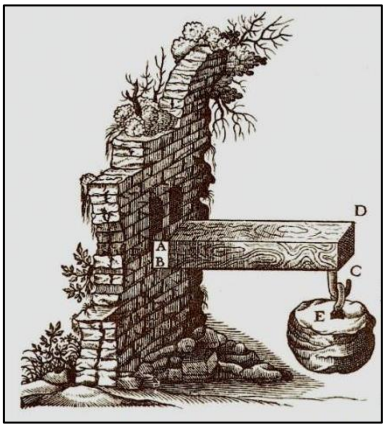


Figure 5. Galileo's *Discorsi,* (p. 157)

He clearly points out the leverage effect at the origin of the observed phenomenon

> *"It is clear that, if the cylinder breaks, fracture will occur at the point B where the edge of the mortise acts as a fulcrum for the lever BC, to which the force is applied; the thickness of the solid BA is the other arm of the lever along which is located the resistance. This resistance opposes the separation of the part BD, lying outside the wall, from the portion lying inside."*

where the "*resistance*" is just the "*absolute resistance to fracture*" already defined, which is the result of uniformly distributed resisting forces as stated in Figure 3 (p. 160)[4].

3. Yield Design of Structures

In both examples presented here above the question to be answered is that of the resistance of a whole body submitted to a prescribed loading process while the

[4] One may wonder about quoting pages that do not seem to be correctly ordered. This comes from the living dialogue style adopted by GALILEO.

resistance of its constituent material is given. The cornerstone of GALILEO's rationale consists in checking more or less accurately that equilibrium of the concerned body or structure is compatible with what he considers as defining the resistance of the material.

Let us focus on the example in Figure 5. The guiding idea in GALILEO's rationale is that the cantilever beam acts as a lever with B as a fulcrum. Then, as schematically represented in Figure 6, he writes down that equilibrium of the structure under the constraint imposed by the resistance of matter requires that the moment at point B of the active load exerted at endpoint C be balanced by the resistance of the rod as defined in Figure 2.

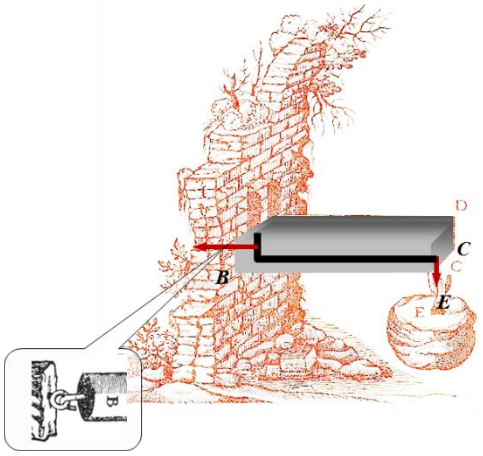


Figure 6. The principle of Galileo's reasoning

As recorded by Henry CREW and Alfonso de SALVIO, GALILEO's rationale for this problem has often been criticised upon many arguments, which we consider do not stay within GALILEO's hypothesis framework. For instance regarding the fact that equilibrium of horizontal forces in the built-in end-section is apparently not satisfied:

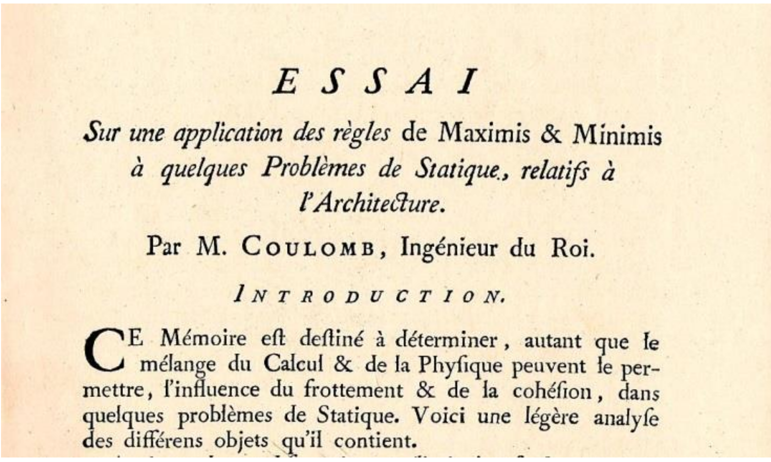
> *"The one fundamental error which is implicitly introduced into this proposition …consists in a failure to see that, in such a beam, there must be equilibrium between the forces of tension and compression over any cross-section."*

As a matter of fact this shortcoming can be explained as the result of GALILEO only considering fracture in tension when defining the resistance of the rod (Figures 2 & 3). Thus the proposed solution amounts to assuming infinite resistance in compression, which would only require one horizontal fibre in B to be in compression to fulfil the horizontal force equilibrium condition without changing the result!

The main point about this example, which is not clearly stated in GALILEO's dialogue, is that the result so-obtained, based upon a necessary condition for the compatibility between equilibrium of the structure and resistance of matter, is just an upper bound value for the applied weight E.

The approach we have just described appears as a prototype for the "stability analyses" that were performed during two centuries by architects and civil engineers when dealing with arches, vaults, cathedrals, churches, bridges, retaining walls either for civilian or military purposes. The reader may refer to the many books and papers by HEYMAN for exhaustive and in depth analyses of this part of the art of building history (http://www.bma.arch.unige.it/PDF/Heyman%20Biblio%20crono.pdf).

Among many authors we must mention COULOMB for his *Essay On the application of the rules of Maximis and Minimis to some Problems of Statics related to Architecture* (Figure 7), whose title recalls LEIBNIZ' seminal contribution to the theory of Calculus (Leibniz, 1684).



ESSAI

Sur une application des règles de Maximis & Minimis *à quelques Problèmes de Statique, relatifs à l'Architecture.*

Par M. COULOMB, Ingénieur du Roi.

INTRODUCTION.

CE Mémoire eſt deſtiné à déterminer, autant que le mélange du Calcul & de la Phyſique peuvent le permettre, l'influence du frottement & de la cohéſion, dans quelques problèmes de Statique. Voici une légère analyſe des différens objets qu'il contient.

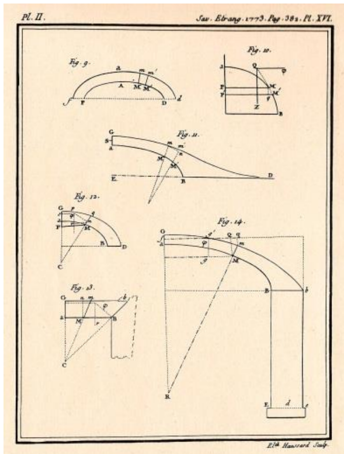


Figure 7. Coulomb's *Essai* (1773)

In that work COULOMB, dealing with stability analyses of vaults, arches… (Figure 7) and active and passive thrust on retaining walls (Figure 8), clearly points out the difference of status between active forces, whose magnitude is imposed, and resisting forces that are only defined through the resistance of matter that sets a limit to their magnitude. Compatibility between equilibrium equations of the concerned system or structure on the one side and resistance of matter on the other side must be checked as a necessary condition for "stability".

But for exceptional cases of statically determinate structures, exhaustive compatibility checking turns out to be unfeasible. The checking process is restrained by substituting weaker necessary equilibrium equations for the complete ones. Most often only global equilibrium equations – or even only one of them – are imposed. It follows, as clearly exposed by COULOMB, that it is necessary to look for the *minimum* value of the load obtained through the procedure.

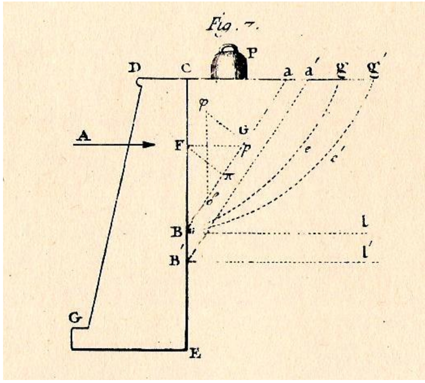


Figure 8. Coulomb's stability analysis of a retaining wall

Moreover, as global equilibrium equations only involve resultant forces and moments they can easily be interpreted from a kinematical viewpoint through the introduction of rigid body motions. Thence, the necessary condition for compatibility between equilibrium and resistance can be written as checking if, in such a motion, the work of active forces can be balanced by the work of resisting forces.

This explains the following two heuristic statements that have been relied upon by designers for quite a long time:

- If an internal force distribution can be built up that is in equilibrium with the given active forces and respect the limits set by the resistance of matter, then the structure will be "stable".
- If a collapse mechanism can be built up where the work by the given active forces exceeds the work developed by resisting forces, then the structure will be "unstable".

Despite their looking rather obvious, both statements happen to be misleading. Examples have been given showing they may even be in contradiction with each other (Drucker, 1954; Salençon, 1974): this means that a set of given active forces that can be proven safe through the first statement is then proven unsafe through the second one.

A reliable mathematical theory was established in the 20th century for such stability analyses, based upon the very same requirement that equilibrium and resistance should be compatible. Works by PRAGER (Prager, 1955) can be considered as seminal on the topic. Translating the old French name "Calcul à la rupture" we call it "Theory of Yield Design" in order to draw a distinction from the Theory of Limit Analysis or Limit Loads, as this latter theory assumes the constitutive law of the constituent material to be fully defined as an Elastic and perfectly plastic law with associated flow rule.

The theory of Yield Design relies on two mathematical concepts (Salençon, 1983, 1990, 2013):

Convexity, for the criteria defining the resistance of the constituent materials,

Duality between stresses and strain rates through the principle of virtual work.

- Underscoring the fact that compatibility between equilibrium and resistance is just a *necessary condition* for the structure to be stable, any set of active loads that can be equilibrated by an internal force distribution complying with the convex strength criterion of matter all over the structure is said to be *Potentially safe*. These potentially safe loads generate a *convex* set K in the loading parameter vector space $(Q_1,...Q_n) = \underline{Q} \in (\mathbb{R})^n$, which opens the way to the *Statical approach* of potentially safe loads.
- Then, the dual definition of the convex set of potentially safe loads results in the *kinematical approach*: any set of active loads whose virtual work in a virtual collapse mechanism exceeds the total maximum resisting work of matter in that mechanism is *Certainly unsafe*[5]

These theorems are mathematically consistent with each other. They look as the counterparts of the preceding heuristic statements, but deserve being explained and clarified.

The wording *Potential stability* recalls that, depending on the actual behaviour of matter before fracture and the complete history of the loading process of the structure, a potentially safe set of active loads may come out as actually safe or not.

In the kinematical approach, *virtual collapse mechanisms* are referred to, with the meaning that they are not compelled to abide with geometrical constraints imposed to the structure or a constitutive law of matter if any, or correspond to observed mechanisms either for fracture of matter or collapse of the structure. Thence the pivotal concept of *Maximum resisting work* is introduced, which is derived through *mathematical duality* from the strength criterion of the constituent material. It is the maximum work that can be developed by internal forces in an element of matter in a

[5] Under some mathematical hypotheses (Frémond & Friaà, 1978) the kinematical approach can also be proven to completely define the convex.

given virtual mechanism under the only constraint set by the convex strength criterion of that element. Within the mathematical framework developed by MOREAU (1967), the maximum resisting work is just the *support function* of the strength criterion, which is at the root of its dual definition. The total maximum resisting work is the result of the integration of that quantity over the whole structure.

As symbolically illustrated in Figure 9, loads and resistances do play symmetric roles in the Yield Design rationale. With the introduction of resistance parameters $(R_1,...R_m) = \underline{R} \in (\mathbb{R}_+)^m$, the theory can be generalised. Potentially safe dimensionings for a given set of loads are defined consequently, which also generate a convex set $D(\underline{Q})$ with similar statical and kinematical approaches. Optimum dimensioning within the Yield Design theory can thus be performed.

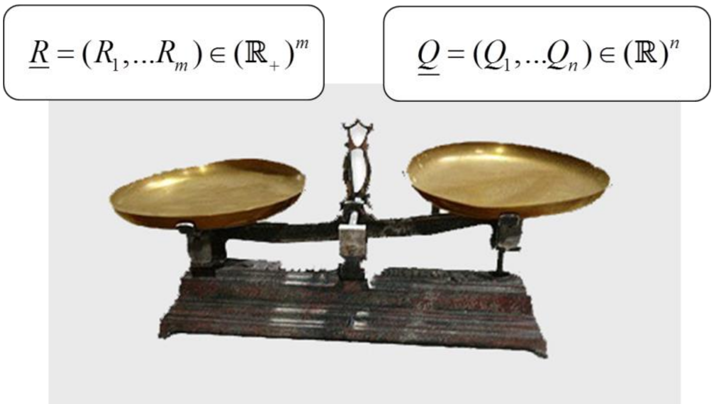


Figure 9. Symmetric roles played by loads and resistances

Then, the more general concept of a *Domain of potential stability* $\mathcal{K}$, referring to both loads and resistances, is defined in $(\mathbb{R})^n \times (\mathbb{R}_+)^m$ in a natural way by

$$(\underline{Q},\underline{R}) \in \mathcal{K} \subset \mathbb{R}^n \times (\mathbb{R}_+)^m \Leftrightarrow \underline{Q} \in K(\underline{R}) \subset \mathbb{R}^n \Leftrightarrow \underline{R} \in D(\underline{Q}) \subset (\mathbb{R}_+)^m : \quad (1)$$

it is a convex cone in $(\mathbb{R})^n \times (\mathbb{R}_+)^m$ which can be determined through the internal approach defined by (1) or the external approach written as

$$\forall \underline{\hat{U}} \text{ virtual collapse mechanism, } \mathcal{K} \subset \left\{ \underline{Q}.\underline{q}(\underline{\hat{U}}) - \underline{R}.\underline{r}(\underline{\hat{U}}) \leq 0 \right\}, \quad (2)$$

where $\underline{q}(\underline{\hat{U}})$ and $\underline{r}(\underline{\hat{U}})$ are respectively the kinematic vector associated with $\underline{Q}$ and the dual vector of $\underline{R}$ in the expression of the virtual resisting work.

This domain is referred to in the case of *Probabilistic* yield design dimensioning (Carmasol, 1983; Carmasol & Salençon, 1985), when the loads or resistances, or both of them are stochastically defined through a probability measure $\mu_{\underline{Q},\underline{R}}$ in $(\mathbb{R})^n \times (\mathbb{R}_+)^m$.

The probability of potential stability of the structure is just the measure of $\mathcal{K}$: $P_{\mathrm{s}}(\mu_{\underline{Q},\underline{R}}) = \mu_{\underline{Q},\underline{R}}(\mathcal{K})$ while the probability of collapse is $P_{\mathrm{c}}(\mu_{\underline{Q},\underline{R}}) = 1 - \mu_{\underline{Q},\underline{R}}(\mathcal{K})$. This latter probability is usually what the designer wishes to assess. It is important to note that equation (2), combined with Monte Carlo methods for instance, only provides lower bounds for $P_{\mathrm{c}}(\mu_{\underline{Q},\underline{R}})$ that may be dramatically optimistic if the range of considered virtual collapse mechanisms is too small. Also, as very low values of $P_{\mathrm{c}}(\mu_{\underline{Q},\underline{R}})$ are anticipated since collapse must be a very rare event, one must pay great attention to the choice of the probability measure $\mu_{\underline{Q},\underline{R}}$ as regards its "tail behaviour".

Owing to its general mathematical formulation, the Yield Design theory applies to the many methods that have been used for long by civil engineers, as mentioned earlier, for the design of vaults, arches, reinforced concrete plates, shells, retaining walls, slope stability analyses in the spirit of the heuristic statements. It also makes it easier to combine these methods when analysing structures that associate different geometrical and mechanical modelling frameworks (e.g. one-dimensional, two-dimensional and three-dimensional continua).

One may think that, due to the fact that the Yield Design approach only provides results in the form of Potential stability, its practical relevance should be rather faint by now, when many numerical methods are available for step-by-step analyses of structures with sophisticated constitutive laws for their constituent materials. Ironically, it is the very reasons that require the introduction of the adverb "potentially" that explain that the Yield Design theorems, most often the kinematical one, are still currently referred to in practical circumstances, especially in civil and construction engineering. As pointed out by HEYMAN (1983, 1988), in the absence of data about the history of the loading process of a structure, e.g. concerning initial internal force distribution (that may be considered as a general rule !) and with rather crude data about the constitutive laws, the Yield Design approach provides an answer to stability analysis problems in the form of upper and lower bounds for the ultimate sets of loads that might be sustained within such a restricted data framework. This unfortunately often happens to be the case with post-collapse analyses.

Also, as illustrated in (Pecker, 2017), these results can be a guiding thread for design projects that will be assessed by various methods numerical and experimental methods, including reduced scale tests.

They also come out as a theoretical groundwork for the "Ultimate Limit State Design", which is one of the philosophies that need being correctly referred to in construction codes. The Theory of Yield design provides clear definitions for the "effect" of design resistances and "effect" of design loads through the kinematical approach, and clarifies the concept of "partial factor for model uncertainties".

Incidentally, reconsidering GALILEO's solution to the problem of the cantilever beam, it comes out that it clearly fits within the mathematical framework of the Theory of Yield Design as an implementation of the kinematical approach, where a rigid body rotation about fulcrum B is used as a virtual collapse mechanism. For this reason the result so obtained remains valid as an upper bound to the weight E whatever the fibre resistance in compression.

4. Dimensional Analysis

We may now come back to GALILEO's heuristic discussion about the size of *"machines"* and try to explain and quantify his arguments in the case of "small" and "big" animals.

Keeping in mind that the concept of internal forces, stress fields for a three-dimensional continuum, would only be introduced two centuries later and, consistently, without any reference to constitutive laws we may look at the problem within GALILEO's own mechanical framework, in the following way.

$\lambda > 1$ is the geometric scale of the big animal with respect to the small one (equal to 3 in the example in Figure 4). Let ρg denote the average animal weight per unit volume for both the big and small animals, whose total weights are respectively $\rho g V$ and $\rho g V_\lambda$. We focus our attention on the vertical bones whose function is to support the total weight of the animal under concern and denote by S_λ and S the total area of their cross sections in the big and small animals respectively, with the corresponding scale factor $\lambda_S = S_\lambda / S$. With σ_0 denoting the resistance of the bone material to compressive force per unit bone section, supposed to be the same for both the small and big animals, we may state that:

- for the *existing* "small" animal the total resistance of those bones is such that

$$\sigma_0 S > \rho g V \tag{3}$$

- and a similar equation must be satisfied for the "big" animal to exist

$$\sigma_0 S_\lambda > \rho g V_\lambda , \tag{4}$$

with $V_\lambda = \lambda^3 V$, from which it is clear that if the bones are to increase harmoniously with the whole body, which implies $S_\lambda / S = \lambda^2$, they will not keep on complying with equation (4) for high values of λ.

If the bone constituent material is unchanged the existence of the big animal requires that the bone geometrical scale factor be increased as $\lambda^{3/2}$, as shown in Figure 4 where $\lambda = 3$ and $\lambda^{3/2} \cong 5.2$.

Otherwise, if the geometrical "harmony" of the animal is to be preserved, the bone constituent material must be stronger, with resistance $(\sigma_0)_\lambda$, in order that

$$(\sigma_0)_\lambda \, S_\lambda > \rho \, g \, V_\lambda \,, \tag{5}$$

as remarked by GALILEO and actually observed in nature. Incidentally, Figure 10 from (Kérisel, 1978), presents the giant footprints of a "big" animal!

Figure 10. "Toro Toro (Bolivia). The gigantic footprints of a dinosaur about 190 million year old. (Centro Studi Richerche Ligabue, Venice)"

With this example it may be considered that GALILEO opened the way to the theory of *dimensional analysis* and the concept of *mechanical similarity* that would be developed more than two centuries later: equation (5) shows that the problem is governed by the dimensionless factor $(\sigma_0)_\lambda \, S_\lambda / \rho \, g \, V_\lambda$.

Presently, dimensional analysis is most often referred to as the use of the Π-theorem, which provides efficient practical methods for exhibiting the non-dimensional factors that govern a physical phenomenon. It thus reduces the bulge of numerical calculations to be performed and makes presentation and analysis of results easier.

Historically, as reported by BORODICH in his contribution to the *IUTAM Symposium on Scaling in Solid Mechanics* (Borodich, 2007), it seems that, after GALILEO's study of the size effect for geometrically similar mechanical structure, EULER also considered the problem of structure modelling (Euler, 1776) and may also be supposed to have been aware of kind of a Π-theorem.

BORODICH also reports a theorem by KIRPICHEV that only concerned weightless elastic models (Kirpichev, 1874). The French mathematician BERTRAND produced the first explicit discussion about dimensional homogeneity (Bertrand, 1878) quite simultaneously and independently of LORD RAYLEIGH's contribution (Rayleigh, 1877, pp. 46-47). In the meantime, FOURIER had already explicitly used the dimension concept (Fourier, 1807, 1819). There seems to be a general consensus (e.g. Macagno, 1971; Debongnie, 2016) about the fact that VASCHY proposed the first form of the Π-theorem which is also known as Vaschy's theorem (Vaschy, 1892). Nevertheless, the theorem is most popular as Buckingham theorem after BUCKINGHAM's paper (Buckingham, 1914), although BUCKINGHAM (Buckingham, 1921) acknowledged having been inspired by the English abstract of a paper in French by RIABOUCHINSKI (Riabouchinski, 1911).

Quoting BARRENBLATT's own wording in the preface of (Barrenblatt, 1987) we may say that

> *"In fact, the idea on which dimensional analysis is based is very simple, and can be understood by all: physical laws do not depend on arbitrariness in the choice of basic units of measurement. An important conclusion can be drawn from this simple idea using simple arguments: the functions that express physical laws must possess a certain fundamental property, which in mathematics is called generalized homogeneity or symmetry. This property allows the number of arguments in these functions to be reduced, thereby making it simpler to obtain them… this is in fact the entire content of dimensional analysis – there is nothing more in it."*

Despite its simplicity, this introduction of the concept of dimensional analysis is followed a few paragraphs further by a *caveat*:

> *"It seems to me that the deficiencies in many of the previous attempts to describe dimensional analysis are explained (strangely enough) by the absence of a logically correct definition of the main concept - the dimensions of a physical quantity. Indeed, the definition of dimensions requires some reflection and the introduction of some additional concepts beforehand."*

As a matter of fact, the mathematical bases of dimensional analysis are not trivial and a full and general mathematical proof has been the object of successive contributions that are listed and commented in (Macagno, 1971; Saint-Guilhem 1962, 1985; Debongnie, 2016;…), where reference is often made to a theorem by FEDERMAN

(Federman, 1911). In more recent papers (Saint-Guilhem, 1985) the use of some results in topology about the properties of arc-wise connected subgroups of $\mathbb{R}^n$ (Hayashida, 1949; Homma & Minagawa, 1949; Bourbaki, 1974), yielded a full mathematical proof of the *One and fundamental theorem of dimensional analysis*.

Going back now to the initial topic of this section, namely the size effect in the modelling of structures, the fact that the dimensionless factor $(\sigma_0)_\lambda S_\lambda / \rho g V_\lambda$ was shown to govern the problem of mechanical similarity is just a particular case of an important and current application of the Π-theorem, which provides clear similarity rules for representative reduced scale experiments.

GALILEO's analysis did anticipate upon a current practice in civil engineering, where reduced scale experiments are commonly used as one of many approaches when designing a structure, especially when it cannot be derived or extrapolated straightforwardly from an existing one due to its size or to specific actions and special constraints.

When designing the foundations of a bridge for instance different materials are involved at the same time. For some of them, whose constitutive law is well characterised and simple enough, it is possible to have different materials corresponding to each other in a proper way in the full-scale structure and in the reduced scale model. But civil engineers also need to cope with soil and similar materials on which their structures are to be safely built and remain stable. Unfortunately the behaviour of these materials is so complex and dependent on the actual level of internal efforts they are submitted to that it is compulsory, for reduced scale experiments to be representative, to keep the material unchanged. Then, gravity forces must be substituted with body forces that are artificially increased in inverse proportion with respect to the scale of the model: the centrifuge modelling technique is commonly used in such cases as illustrated in Figure 11 (Garnier, J.& Pecker, A., 1999), as well as others such as the hydraulic gradient. We may infer that the experimental difficulties COULOMB had to face when, as an Officer of the Royal Corps of Engineers in Martinique (West Indies) he was elaborating his theory about the thrust of soil on retaining walls, were due to this necessity that was not realised at the time; as quoted by KÉRISEL (Kérisel, 1987, p. 57):

> *"I have often come across situations in which all the theories based on hypotheses or on small-scale experiments in a physics laboratory have proved inadequate in practice: I did all the necessary research…"*

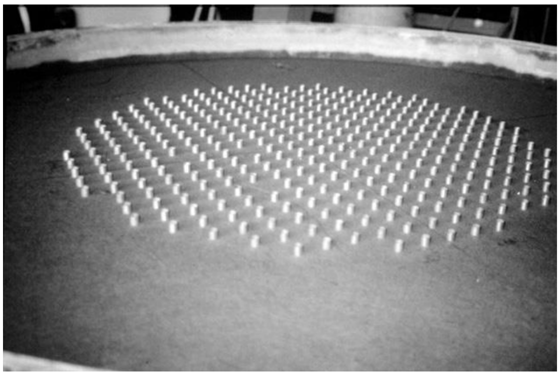


Figure 11. Centrifuge test on a reduced scale model of a reinforced soil foundation for the Rion-AntiRion Bridge subject to seismic forces

5. As a conclusion

We hope that these two examples made it clearer that Galileo's insight was at the origin of theoretical research and practical applications that are still actively referred to presently. Through mathematical concepts introduced during the past century complete proofs and statements of what could have been somehow considered as heuristic approaches could be established that ascertained the corresponding results within a clear and appropriate framework.

The nexus between materials and structures, whatever their size, is a most important topic for Solid mechanicists and engineers. Albeit it should not be considered as the unique goal, improving material resistance, as observed by GALILEO, often comes out as a necessity. As a recent example we may mention the stupendous results obtained last year by IAS Senior Fellow Professor LU Jian and co-workers about a new method for producing high-strength magnesium alloys (Wu & al., 2017).

6.References

BARENBLATT, G. I. (1987) – *Dimensional Analysis*. Gordon & Breach Sc. Publ., New York.

BERTRAND, J. (1878) – Sur l'homogénéité dans les formules de Physique. *Comptes Rendus Académie des sciences,* Paris, **86**, 916-920. http://gallica.bnf.fr/ark:/12148/bpt6k3043m/f5.item.r=J.

BORODICH, F. M. (2007) – Scaling Transformations in Solid Mechanics. *IUTAM Symposium on Scaling in Solid Mechanics*, Cardiff, UK, 25-29 June 2007, Springer, 2009, pp. 11-26.

BOURBAKI, N. (1974) – *Topologie générale*. Chap. VII, §1, Ex. 16.

BUCKINGHAM E. (1914) – On physically similar systems: illustrations of the use of dimensional analysis. *Phys. Rev.* **4**, 354-377.

BUCKINGHAM E. (1921) – Notes on the method of dimensions. *Phil. Mag.*,**42**, 696-719. https://www.tandfonline.com/doi/abs/10.1080/14786442108633812

CARMASOL, A. (1983) – *Approche probabiliste du calcul à la rupture.* Doctor-Engineer Thesis, École nationale des ponts et chaussées, Paris.

CARMASOL, A. & SALENÇON, J. (1985) – Une approche probabiliste du dimensionnement des structures par le calcul à la rupture. *J. Mec. Th. Appl.*, **4**, 3, 305-321.

COULOMB, C-A. (1773) – *Essai sur une application des règles de Maximis et Minimis à quelques problèmes de statique relatifs à l'architecture.* Mémoire présenté à l'Académie royale des sciences, Paris.

DEBONGNIE, J. F. (2016) – *Sur le théorème de Vaschy-Buckingham*. http://hdl.handle.net/2268/197814

DRUCKER, D.C. (1954) – Coulomb friction, plasticity and limit loads. *Jl. Appl. Mech. Trans. ASME*, **21**, 1, 71-74.

EULER, L. (1776) – Regula facilis pro dijudicanda firmitate pontis aliusve corporis similis ex cognita firmitate moduli. *Novi Commentarii Academiae scientiarum Imperialis Petropolitanae*, **20**, 271-285.

FEDERMAN, A. (1911) – On some general methods of integration of partial differential equations of the 1st order (in Russian). *Proceedings of the Saint-Petersburg polytechnic institute. Section of technics, natural science, and mathematics)*, **16**, IX & X, 97-155.

FOURIER, J. (1807) – *Mémoire sur la théorie de la chaleur*. Manuscrit N°267, Bibliothèque de l'École des ponts et chaussées, Paris.

FOURIER, J. (1819) – Théorie du mouvement de la chaleur dans les corps solides. *Mémoires de l'Académie Royale des sciences*, vol. IV, Paris.

FOURIER, J. (1822) – *Théorie Analytique de la Chaleur.* Didot, Paris.

FRÉMOND, M. & FRIAÀ, A. (1978) – Analyse limite. Comparaison des methods statique et cinématique. *Comptes rendus Académie des sciences de Paris,* **286**, 107-110.

GALILEO GALILEI (1638) – *Dialogues Concerning Two new Sciences.* Translation by Henry CREW & Alfonso de SALVIO (1914), Dover publications Inc., New York (1954).

GALILEO GALILEI (1638) – *Discorsi e Dimostrazioni Matematiche intorno à due nuove scienze*. Elsevirii, Leyden.

GARNIER, J. & PECKER, A. (1999) – Use of Centrifuge Tests for the Validation of Innovative Concepts in Foundation Engineering. *Proceedings 2nd International Conference on Earthquake Geotechnical Engineering*, Lisbon, 21-25 June 1999, Balkema, 433-439.

HAYASHIDA, T. (1949) – Arc-wise connected subgroup of a vector group. *Kodai Mathematical Seminar Reports*, **1**, 16-17.

HEYMAN, J. (1983) – Chronic defects in masonry vaults: Sabouret's cracks. *Momentum*, **26**, 131-141.

HEYMAN, J. (1988) – Poleni's problem. *Proc. Instn Civ. Engrs*, 1, **84**, 737-759.

HOMMA, T. & MINAGAWA, T. (1949) – Vector Group in Real Euclidean Space. *Kodai Mathematical Seminar Reports*, **1**, 19-20.

KIRPICHEV, V.L. (1874) – On similitude atelastic phenomena. *Jl Rus. Chem. Soc. Phys. Soc. Phys. Part. Div. I*, **6** (8), 152-155.

KÉRISEL, J. (1985) – *Down to Earth*. Balkema, Rotterdam, Boston.

LEIBNIZ, G. W. (1684) – Nova Methodus pro Maximis et Minimis. *Acta Eruditorum*, Leipzig, 467-473.

MACAGNO, E. O. (1971) – Historico-critical Review of Dimensional Analysis. *Jl of the Franklin Institute*, **299**, 6, 391-402.

https://www.researchgate.net/publication/239368382_Historicocritical_review_of_dimensional_analysis

MOREAU, J-J. (1967) – *Fonctionnelles convexes.* Séminaire sur les équations aux dérivées partielles, Collège de France, Paris.

PECKER, A. (2017) – *Research and Practice in Engineering: The Rion-Antirion Bridge Foundation Design.* IAS Distinguished lecture, City University, Hong Kong, November 15, 2017.

PRAGER, W. (1955a) – Théorie générale des états limites d'équilibre. *J. Math. Pures et Appliquées.*, **34**, 4, 395-406.

PRAGER, W. (1974) – Limit analysis: the development of a concept. *Problems of Plasticity.*, SAWCZUK, A. ed., Nordhoff International, Leyden, The Netherlands, pp. 3-24.

RAYLEIGH, J. W. S. (1877) – *The theory of sound*. MacMillan, pp. 46-47; Dover, New York, 1945. http://gallica.bnf.fr/ark:/12148/bpt6k951307?rk=42918;4.

RAYLEIGH, J. W. S. (1892) – On the question of the stability of fluids. *Phil. Mag.*, **34**, 59-70.

RIABOUCHINSKI, D. P. (1911) – Méthode des variables de dimension zéro, et son application en aérodynamique. *L'Aérophile*, **1**, septembre 1911, 407-408.

SAINT-GUILHEM, R. (1962) – *Les principes de l'analyse dimensionnelle, invariance des relations vectorielles dans certains groupes d'affinités*. Mémorial des sciences mathématiques, **152**, Gauthier-Villars, Paris. http://www.numdam.org/issue/MSM_1962__152__3_0.pdf.

SAINT-GUILHEM, R. (1985) – Sur les fondements de la similitude physique: le théorème de Federman. *J. Mec. Th. & Appl.*, **4**, 3, 337-356.

SALENÇON, J. (1974) – Un exemple de non-validité de la théorie classique des charges limites pour un système non standard. *Int. Symp. on Foundations of Plasticity*, Warsaw, 1972. *Problems of Plasticity*, SAWCZUCK, A. ed. Noordhoff, Leyden, 1974, pp. 432-439.

SALENÇON, J. (1983) – *Calcul à la rupture et analyse limite*. Presses de l'École nationale des ponts et chaussées, Paris.

SALENÇON, J. (1990) – An introduction to the Yield Design Theory and its applications to soil mechanics. *Eur. J. Mech. A/Solids,* **9**, 477-500.

SALENÇON, J. (2013) – *Yield Design*. ISTE-Wiley, 240 p.

SALENÇON, J. (2018) – *Virtual Work Approach to Mechanical Modeling*. ISTE-Wiley, 335 p.

VASCHY, A. (1892). – Sur les lois de la similitude en Physique. *Annales télégraphiques,* **19**, 25-28.

VOLTAIRE (AROUET DIT), F. (1737) – *Sixième discours en vers sur l'homme*.

WU, G., CHAN, K-C., ZHU, L. & LU, J. (2017) – Dual-phase nanostructuring as a route to highstrength magnesium alloys. *Nature*, **545**, 80-83, doi:10.1038/nature21691.