

EQUILIBRIUM AND FORCES: FROM ARISTOTLE TO LAGRANGE

1 IN BRIEF

The historical path to LAGRANGE's statement of the Principle of virtual velocities has been two-millennium long, a facet of what BENVENUTO calls "*The Enigma of Force and the Foundations of Mechanics*" and could "*be regarded as vague meandering, impotent struggles, foolish attempts at reduction, and justified doubt about the nature of force*" (TRUESDELL¹). Many authors, after LAGRANGE (1788) himself, have tracked the history and avatars of the concepts of virtual velocities and virtual work: a very comprehensive analysis appears in (Duhem, 1905, 1906) and we must also quote, among others, (Dugas, 1950), (Truesdell, 1968), (Benvenuto, 1981, 1991), (Capecchi, 2012)...

The purpose of this contribution is just to present some milestones along the historical path up to LAGRANGE's contribution.

2 THE "SIMPLE MACHINES"

When looking for the very roots of Mechanics one inescapably encounters the study of the "simple machines" that provide mechanical advantage, or leverage, when applying a single active force to do work against a single load force, such as the weight of a body: "*Early theoretical thinking about statics and mechanics took as its references particular objects, things like the lever, used since ancient times as necessary tools.*"². ARISTOTLE's (384-322 BC) *Quaestiones Mechanicae* (Mechanical Problems)³, as quoted by BENVENUTO⁴, defines Mechanics as an Art:

"Miraculously some facts occur in physics whose causes are unknown; that is, those artifices that appear to transgress Nature in favour of man... Thus, when it is necessary to do something that goes beyond Nature, the difficulties can be overcome with the assistance of art. Mechanics is the name of the art that helps us over these difficulties; as the poet Antiphon put it, 'Art brings the victory that Nature impedes.'"

Regarding the lever problem, he finds the overruling marvellous explanation in the circles being described by the weight and the "small force" with different velocities:

¹ Foreword to (Benvenuto, 1991), p. IX.

² (Benvenuto, 1991), p.4.

³ Apocryphal?

⁴ (Benvenuto, 1991), p.XVIII.

“Among the problems included in this class are included those concerned with the lever. For it is strange that a great weight can be moved by a small force, and that, too, when a greater weight is involved. For the very same weight, which a man cannot move without a lever, he quickly moves by applying the weight of the lever. Now the original cause of all such phenomena is the circle; and this is natural, for it is in no way strange that something remarkable should result from something more remarkable, and the most remarkable fact is the combination of opposites with each other. The circle is made up of such opposites, for to begin with it is composed both of the moving and of the stationary, which are by nature opposite to each other.”... “Therefore, as has been said before, there is nothing strange in the circle being the first of all marvels.”

“...Again, no two points on one line drawn as a radius from the centre travel at the same pace, but that which is further from the fixed centre travels more rapidly; it is due to this that many of the remarkable properties in the movement of circles arise.”

ARISTOTLE’S *Physicæ Auscultationes* (Physics) are usually referred to for the introduction of the concept of (motive) “Power” [δύναμις or ἰσχύς] representing the product of the weight of the considered body by its velocity (the ratio of the displacement to the duration of the movement) (De Groot, 2008) in order to explain the principle of the rectilinear lever. The equilibrium of the lever is just stated as the equality (equivalence) of the powers acting at each end, explaining the mechanical advantage by the comparison of the velocities of the active and load forces. The “*rule of proportion*” (Physics, vol. VII, chap. V) is then stated as follows, with a major ambiguity due to his reference to time:

“Then, the movement A have moved B a distance G in a time D, then in the same time the same force A will move ½ B twice the distance G, and in ½ D it will move ½ B the whole distance for G: thus the rules of proportion will be observed.”

This ambiguity was only to be definitely ruled out by DESCARTES (§4.1).

ARCHIMEDES’ (287-212 BC) approach of statics in *De Planorum Æquilibriis* (On the equilibrium of planes) is completely different: “*While Aristotle relates mechanics to a physical theory, aiming for a universal synthesis, Archimedes thinks of statics as a rational and autonomous science, founded on almost self-evident postulates and built upon rigorous mathematical demonstrations*”⁵.

A very illustrative example is given by his proof of the law of the lever (or steelyard) that can be sketched as follows (Figure 1). The initial accepted demand is that a lever with arms of equal length ($p + q$) over which the load – weight – is uniformly distributed

⁵ (Benvenuto, 1991), p. 43.

is in equilibrium (a). Then, through a *thought experiment* (b), this lever is split into two parts of length $2p$ and $2q$ respectively (and the corresponding loads) which, anticipating on the terminology we will use in the following chapters, we may call *subsystems* of the given physical *system* (the lever). Considering first the subsystem with length $2p$ we can state, from the same initial demand, that it is in equilibrium about its midpoint Q where it exerts the load $2p$; in the same way the subsystem with length $2q$ is in equilibrium about its midpoint P where it exerts the load $2q$. These midpoints are respectively at a distance q and p from the midpoint O of the lever: thus the equilibrium of the whole lever with equal arms is also the result of the equilibrium about O of the lever QP with unequal arms q and p , and loads $2p$ and $2q$ respectively (c).

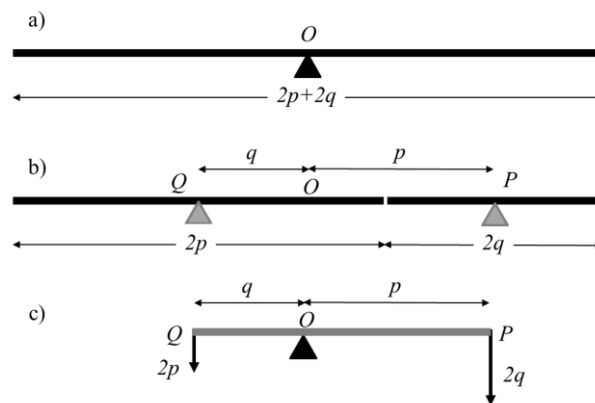


Figure1 – Archimedes’ proof of the law of the lever

This proof based upon a statical thought experiment does not call upon any overruling general principle involving time or velocity. It has been discussed by many authors and various “improvements” were put forward that are listed and analysed in (Benvenuto, 1991) and won’t be discussed in this brief outline whose purpose is to introduce the two fundamental pathways that were to be followed all along the History of Mechanics. Schematically one could say that ARCHIMEDES aimed at providing answers to given practical problems based upon a limited number of preliminary demands, while ARISTOTLE would try to formulate a general principle, in the present case the equality of the powers of the active force and load force, to cope with any possible problem.

Making use of DESCARTES’s very words⁶ in his criticism of GALILEO’s analyses of the steelyard and lever, DUHEM⁷ wrote that ARCHIMEDES “plainly explains *Quod ita sit* but not *Cur ita sit*” (“What” but not “Why”) and commented ARISTOTLE’s analysis:

“This insight is, indeed, the seed from which will come out, through a twenty century development, the powerful ramifications of the Principle of virtual velocities”.

⁶ (Descartes, 1668), *Correspondance* 2, p. 433.

⁷ (Duhem, 1905) p. 332.

A similar comment had been made by FOURIER in his *Mémoire sur la Statique* (A Memoir on Statics)⁸:

“One may add that his writings offer the first insights on the Principle of virtual velocities”

3 LEONARDO, STEVIN, GALILEO

It is clear that we are still very far from a general statement of the principle. The concepts need being extracted as an essence through a long lasting trial-and-error process that cannot be presented extensively here following the historical time line and quoting all contributions that are reported in the analyses of DUHEM, DUGAS, BENVENUTO, CAPECCHI.... The story will be made short.

Among the many topics he covered in his manuscripts, which are stored and preserved in the Library of the *Institut de France* in Paris⁹, LEONARDO (1431-1519) detailed the properties of the simple machines (*Ms. A, E, F, I and M*) and tried to express them through a simple general law that turns out to be quite similar to ARISTOTLE's statement (*Ms. F*):

“If a power [puissance] moves a given body along a given length of space during a given time span, it will move half this body during the same time span along twice the given length of space. Or the same power [vertu] will move half this body along the same length of space in half the same time span”.

Simon STEVIN (1548-1620), sometimes called Stevinus, was a Flemish-Dutch-Netherlandish mathematician, physicist and engineer. As a mechanist, referring to the lever problem, he discarded ARISTOTLE's argument referring to velocities along the circular trajectories with the simple, hammer like, statement that (Stevin, 1605/1608)

“What is immobile does not describe circles, but two weights in equilibrium are immobile; thus two weights in equilibrium do not describe circles”¹⁰

which actually underscores a true conceptual difficulty: why should the equilibrium of a system be studied by referring to motion?

For the analysis of the inclined plane, STEVIN derived the condition for the balance of forces using a diagram with a "wreath" or necklace containing evenly spaced round balls resting on a triangular wedge (Figure 2). He concluded that if the weights were not proportional to the lengths of the sides on which they rested they would not be in equilibrium since the necklace would be in perpetual motion, which he considered

⁸ (Fourier, 1797).

⁹ (Leonardo Da Vinci, 1487-1508) – *Les Manuscrits de Léonard de Vinci*. Ms A-M.

¹⁰ (Benvenuto, 1991), p.82, cf. also (Duhem, 1905), p.267.

obviously impossible. Incidentally, STEVIN was so proud of his proof that the corresponding Figure appears in the cover of his books *De Beghinselen der Weeghconst* (Stevin, 1586/1605) and *Hypomnemata Mathematica* (Stevin, 1605/1608) with his motto “*Wonder en is gheen wonder*” (“Magic is no magic” also translated by “Wonder, not miracle”)¹¹ as a refutation of ARISTOTLE’s “marvel”. But one may wonder whether this “proof” wasn’t, as a matter of fact, a kind of thought kinematical experiment?

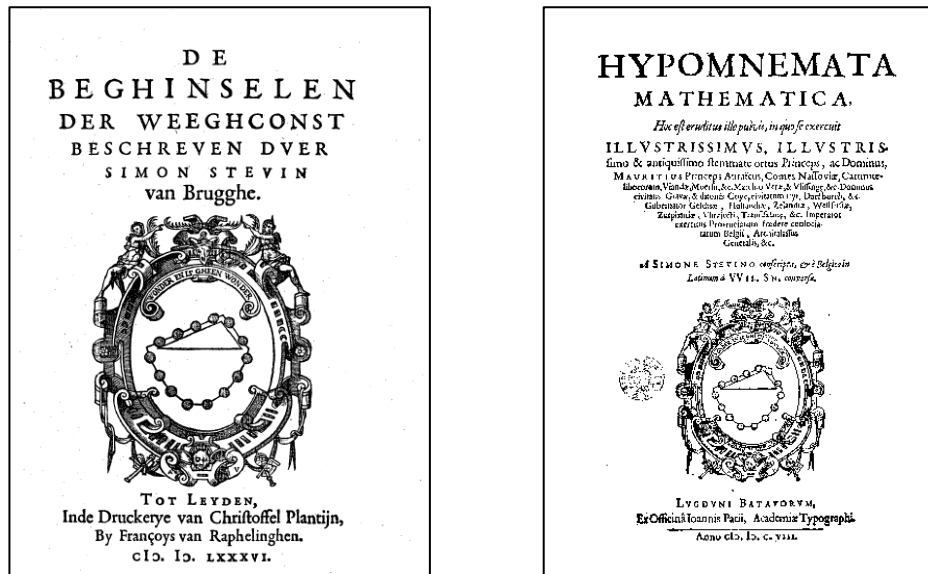


Figure 2 – Covers of STEVIN’s books ¹²

Also, in *Hypomnemata Mathematica*, when dealing with pulleys and pulley blocks STEVIN wrote the following remark

“*Ut spatium agentis, ad spatium patientis:*

“*Sic potentia patientis ad potentiam agentis*”

that may be translated as

“*As the space of the actor is to the space of the sufferer,*

“*So is the power of the sufferer to the power of the actor*”¹³,

and expresses that the displacement of the resistance is to the displacement of the power as the power to the resistance. We observe that there is no reference to time in this sentence that sounds like a *rule of proportion*. According to BENVENUTO, STEVIN would not give this statement the status of a principle, which he disliked: he considered it “as a criterion, not an explanation of equilibrium”¹⁴.

¹¹ Sometimes called “The epitaph of Stevinus”.

¹² The Elements of the Art of Weighing.

¹³ (Benvenuto, 1991), p. 81, where this principle is said to have been already stated by Guidobaldo dal Monte.

¹⁴ (Benvenuto, 1991), p. 82.

In GALILEO's (1564-1642) works (Galileo, 1599, 1634, 1638) one encounters several occurrences of an implicit use of a concept close to what would be defined later on as virtual work. A famous example is related to the analysis of the inclined plane in *Della Scienza Meccanica*¹⁵:

“...Thence the weight F moves downwards, drawing the body E on the sloped plane, this body will cover a distance along AC equal to the one described by the weight F in its fall. But this should be observed: it is true that the body E will have covered all the line AC in the time the weight F falls down an equal length; but during this time, the body E will not have moved away from the common centre of weights more than the vertical length BC, while the weight F, falling down according to the vertical, has dropped a length equal to all the line AC. Recall that weights only resist an oblique motion inasmuch as they move away from the centre of the Earth... We can thus say rightly that the travel [viaggio] of the force [forza] F is to the travel [viaggio] of the force [forza] E in the same ratio as the length AC to the length CB.”

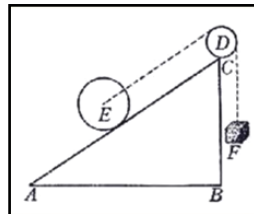


Figure 3 – The inclined plane, (Galileo, 1599)

Despite the fact that this proof was only based upon the concomitant displacements or travels [viaggi] of the weight and the body with respect to the “common centre of weights” GALILEO, as a foreword, still referred to time and velocity in the Aristotelian spirit:

“Finally, let us not overlook the following consideration: as a principle, we said that necessarily, in any mechanical instrument, as much the force was increased via this instrument, as much, on the other hand, one would lose time or velocity.”

GALILEO did actually cope with the difficulty we mentioned earlier about analysing equilibrium through the consideration of motion as shown, for instance, in his analysis of the bearing capacity of a cantilever beam (Galileo, 1638) where he almost introduced the concept of virtual motion in a thought-experiment for designing the rectangular lever that would model the built-in beam (Figure 4) and be analysed from the statical viewpoint, and clearly addresses the concept of *resistance*:

¹⁵ (Galileo, 1634).

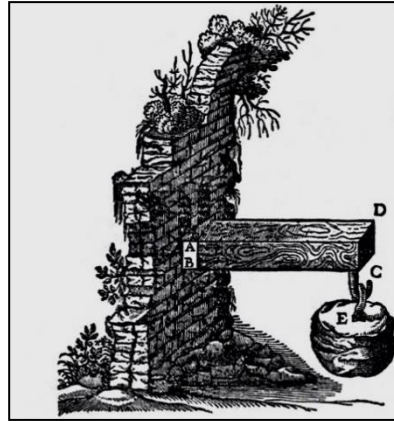


Figure 4 – The cantilever beam, (Galileo, 1638)

“...It is clear that, if the cylinder breaks, fracture will occur at the point B where the edge of the mortise acts as a fulcrum for the lever BC, to which the force is applied; the thickness of the solid BA is the other arm of the lever along which is located the resistance...”

4 DESCARTES AND BERNOULLI

4.1 RENÉ DESCARTES (1596-1650)

The correspondence of DESCARTES (1596-1650)¹⁶, as published by ADAM and TANNERY shows, through many examples, that he had a much clearer vision of a virtual velocity principle than his predecessors or contemporaries whom he would sometimes treat rather roughly as in a letter to MERSENNE (November 15, 1638):

*“Pour ce qu'a écrit Galilée touchant la balance et le levier, il explique fort bien quod ita sit, mais non pas cur ita sit, comme je fais par mon Principe. Et pour ceux qui disent que je devois considerer la vitesse, comme Galilée, plutot que l'espace, pour rendre raison des Machines, je croy, entre nous, que ce sont des gens qui n'en parlent que par fantaisie, sans entendre rien en cette matiere....”*¹⁷

“Regarding what Galileo wrote about the steelyard and the lever, he plainly explains what happens but not why it happens, as I do it myself through my Principe. And as for those who pretend that I should consider velocity, as Galileo does, instead of space, I believe, between us, that they are just people who talk without any understanding of the matter at hand”.

His principle, he had stated it plainly, answering a letter from Constantijn HUYGENS (Christian's father) on October 5, 1637 about the fundamental principle of the simple machines in its common form:

¹⁶ (Descartes, 1668).

¹⁷ (Descartes, 1668). *Correspondance* 2, p. 433.

“The invention of all these machines is founded on one principle, which is that the same force which can lift a weight, for example of 100 pounds, up to two feet, can also lift a weight of 200 pounds up to one foot, or a weight of 400 pounds up to half a foot...”

which expresses the conservation of the product of the load by its vertical displacement and looks like a reminder of ARISTOTLE’s rule of proportion without any reference to time. In the following of the letter he examined such simple machines as the pulley, inclined plane, wedge, etc. within this framework. Nevertheless one point still needed to be corrected and later on, in a letter to MERSENNE (July 13, 1638), DESCARTES insisted on the fact that the displacements to be considered were infinitesimal:

“From this it follows evidently that the gravity relative to a given body, or equivalently the force to be exerted to sustain it or prevent it from going down, when it is in a given position, should be measured by means of the beginning of the movement that would be done by the power which sustains it either for lifting it or following it if it went down.”

which is obviously a major step forward as regards the final formulation of the principle. In order to make himself clearer he added, a few lines below:

“Note that I say begin to go down and not simply go down, because it is only the beginning of the descent that must be taken into account”.

Apart from the infinitesimal character of the displacements that must be considered, one should note that DESCARTES in the French wording makes use of the conditional or potential mode for the verb “*par le commencement du mouvement que devoit [devrait] faire la puissance qui le soustient [soutient]*”. This opens the way to the concept of virtuality of these displacements and also counters STEVIN’s objection about the contradiction between equilibrium and motion. Finally, let us quote a short sentence in a letter which is usually considered as having been sent to BOSWELL in 1646, where DESCARTES discarded actual velocities as the cause of the properties of such simple machines as the lever in the Aristotelian way:

“I do not deny the material truth of what Mechanicists usually say, namely that the higher the velocity of the longer arm of the lever compared with the shorter arm, the smaller the force necessary to move it; but I do deny that velocity or slowness be the cause of this effect.”

In other words, referring to time or velocities is not erroneous but just irrelevant.

DESCARTES' fundamental statement was generalized by WALLIS (1616-1703)¹⁸, dealing with any kind of forces with the proper definition of their forward or backward movements:

“And, as a general rule, the forward or backward movements of motor forces whatsoever [virium motricium quarumcunque] are obtained from the products of the forces by their forward or backward movements estimated along the directions of these forces”.

4.2 JOHANN BERNOULLI (1667-1748)

Thanks to Pierre VARIGNON in his *Nouvelle Mécanique ou Statique*,¹⁹ we have the exact wording of the letter Johann BERNOULLI sent him on January 26, 1717. In this letter BERNOULLI gave the first definitions of the concepts of *Energy* [Énergie] and *Virtual Velocities* [Vitesses virtuelles] in the case of a small rigid body motion. Defining Virtual Velocities, he considers small rigid body movements and the components of the corresponding small displacements of the forces along their lines of action:

“Imagine several different forces which act according to various trends or directions to maintain a point, a line, a surface or a body in equilibrium. Imagine that a small movement, either parallel to any direction or about any fixed point be imposed to all this system of forces. It will be easy for you to understand that in this movement each of these forces advances or moves back in its direction, unless one or more of these forces have their own tendencies [tendances] perpendicular to the direction of the small movement; in which case this force or these forces, would not advance nor move back; because these movements forward or backward, which are what I call virtual velocities, are just what the quantities in which each tendency line increases or decreases in the small movement.”

He then defines the *Energy* of each force as the product of the considered force by its virtual velocity in the movement, either “affirmative” or “negative” depending on whether the force moves forward or backward. As a matter of fact this is just the definition of the work by the considered force in the small displacement of its point of application, a concept that had not been introduced before but by SALOMON DE CAUS (1576-1630) with the French word “*Travail*” and its present meaning in his book *Les raisons des forces mouvantes* (Salomon de Caus, 1615).

With these definitions BERNOULLI issues the general statement that

“For any equilibrated system of forces...the sum of the affirmative energies will be equal to the sum of negative energies counted positive”.

¹⁸ (Wallis, 1670).

¹⁹ (Varignon, 1725), II, ix, p. 176.

In the 2nd volume of *Les Origines de la Statique*²⁰, DUHEM could not help lamenting that BERNOULLI adopted the terminology “*vitesses virtuelles*” [virtual velocities] instead of virtual displacements since time and velocities have nothing to do in that matter, and also that this terminology had been retained by many authors. As a response to that criticism one may argue now that this terminology makes it impossible to forget about the infinitesimal character of the quantities involved. The word *virtual* qualifying those velocities may be considered sufficient to recall that they are no velocities at all but just test functions in the mathematical sense of functional analysis. It could be suggested that the term “virtual velocities” be understood as a whole to name these functions.

5 LAGRANGE (1736-1813)

5.1 LAGRANGE’S DEFINITION OF THE FORCE CONCEPT

Up to this point, reading the statements we have reproduced, we implicitly assigned to the word “force” the meaning we give it today but it must be understood that the corresponding concept was still to receive a plain definition as in LAGRANGE’S *Mécanique analytique*²¹:

“On entend, en général, par force ou puissance la cause, quelle qu’elle soit, qui imprime ou tend à imprimer du mouvement au corps auquel on la suppose appliquée ; et c’est aussi par la quantité du mouvement imprimé, ou prêt à imprimer, que la force ou puissance doit s’estimer. Dans l’état d’équilibre, la force n’a pas d’exercice actuel ; elle ne produit qu’une simple tendance au mouvement ; mais on doit toujours la mesurer par l’effet qu’elle produirait si elle n’était pas arrêtée.”

That, we may translate as follows:

“We generally mean by force or power the cause, whatever it is, which imparts or tends to impart a movement to the body to which it is supposed to be applied; and it is also by the quantity of the movement imparted, or ready to be imparted, that the force or power must be estimated. In an equilibrium state, the force does not have a current exercise; it only produces a simple tendency to movement; but one must always measure it by the effect it would produce if it were not stopped.”

Thence, echoing GALILEO’S and NEWTON’S laws of inertia (Newton, 1687), the concept of force appears as the abstract cause of the alteration of motion it is actually imparting or would potentially tend to impart to a body. This last point is especially relevant when dealing with Statics, where only “tendencies” can be considered.

²⁰ (Duhem, 1906), footnote p. 268.

²¹ (Lagrange, 1788).

5.2 LAGRANGE'S STATEMENT OF THE PRINCIPLE

LAGRANGE then gave his consistent definition of virtual velocities:

*“One must understand by virtual velocity, the velocity which a body in equilibrium would be ready to receive, in case this equilibrium should be upset; i.e. the velocity that this body would really take in the first instant of its movement.”*²²

and his generalised statement of the principle of virtual velocities²³:

“If a system of bodies or points, each of them being submitted to arbitrary powers, is in equilibrium, and if this system is given a small unspecified movement, in which each point moves along an infinitely small distance, which is its virtual velocity, the sum of the products of each power by the distance travelled by its point of application along the direction of that power, will always be equal to naught, with the small distances being counted positive when they are travelled in the direction of the power and negative in the opposite direction.”

What is most important in this statement is that it explicitly deals with a *system* of bodies or points. It introduces “a small *unspecified* movement of the system”, defined by independent small *unspecified* movements of each point of the system as in LAGRANGE'S own proof of the principle given hereunder.

5.3 LAGRANGE'S PROOF OF THE PRINCIPLE

A few years later, in a paper published in the *Journal de l'école polytechnique*²⁴, LAGRANGE expressed his dissatisfaction as to the principle of virtual velocities being usually derived from the principles of composition of forces and equilibrium of the lever which he considered not evident enough to be taken as a basis. He thus presented a new proof based upon the pulley block equilibrium principle.

The main thrust of the reasoning is to consider that the forces applied to each body of a system (as in the preceding references, the term “body” refers to a material point) are exerted by means of a weight acting at one end of an ideally inextensible, flexible and weightless string through as many fixed and mobile pulley blocks and tackles as necessary, the other extremity of the string being fixed. Then the argument proceeds from

²² (Lagrange, 1788): “On doit entendre par *vitesse virtuelle* celle qu'un corps en équilibre est disposé à recevoir, en cas que l'équilibre vienne à être rompu, c'est-à-dire la vitesse que ce corps prendrait réellement dans le premier instant de son mouvement”.

²³ (Lagrange, 1788): “Si un système quelconque de tant de corps ou points que l'on veut, tirés chacun par des puissances quelconques, est en équilibre, et qu'on donne à ce système un petit mouvement quelconque, en vertu duquel chaque point parcourt un espace infiniment petit qui exprimera sa vitesse virtuelle, la somme des puissances, multipliées chacune par l'espace que le point où elle est appliquée parcourt suivant la direction de cette même puissance, sera toujours égale à zéro, en regardant comme positifs les petits espaces parcourus dans le sens des puissances, et comme négatifs les espaces parcourus dans un sens opposé.”

²⁴ (Lagrange, 1797), p. 115-118.

the statement²⁵ that equilibrium of the system is reached in the state where any infinitely small displacement of each body does not produce any downward movement of the active weight at the end of the string. It must be noted that no geometrical constraints, either internal or external, are imposed on the bodies of the system.

More precisely, using the notations LAGRANGE adopted in the subsequent editions of the *Mécanique analytique* where he reproduced this approach, we try to illustrate this description in Figure 5 (as a rule, LAGRANGE did not provide any figure: “You will not find Figures in this Work. The methods I use require neither constructions nor geometrical or mechanical arguments, but only algebraic operations, in a regular and uniform course”²⁶). For simplicity, we consider the simple case of two bodies (points). One is connected to three pulley-blocks where the numbers of pulleys are $P/2$, $Q/2$ and $T/2$ respectively with P , Q , T even integers. In the same way, the other body is connected to two pulley-blocks with $R/2$ and $S/2$ pulleys (R and S even integers).

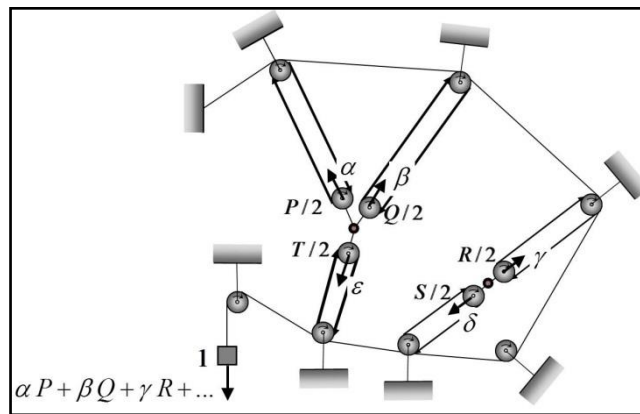


Figure 4 – Our attempt to illustrate Lagrange’s proof of the principle of virtual velocities

As a consequence of the pulley-block equilibrium principle, the forces acting on the bodies are commensurable and, taking the active weight at the extremity of the string as a unit, their values are P , Q , T along the corresponding directions of the string for the first body, R and S for the second one.

From this description of the system loading process, LAGRANGE derives as an obvious condition that in the equilibrium state any arbitrary infinitesimal displacement of each body (point) about its equilibrium position produces no downward movement of the weight at the free extremity of the string. This condition can be expressed explicitly as follows.

²⁵ A “demand” as in Archimedes’ rationales?

²⁶ “On ne trouvera point de Figures dans cet Ouvrage. Les méthodes que j’y emploie ne demandent ni constructions, ni raisonnements géométriques ou mécaniques, mais seulement des opérations algébriques, assujetties à une marche régulière et uniforme.”

Infinitesimal arbitrary displacements of the bodies result in the distances between the mobile pulley blocks and the corresponding fixed ones being reduced (algebraically) by the quantities $\alpha, \beta, \gamma, \dots$ respectively. As a consequence, taking in consideration the number of pulleys in each pulley-block, the free extremity of the string will move downward along the infinitesimal distance $P\alpha + Q\beta + R\gamma + \dots$. Writing down the equilibrium condition as stated here above, *i.e.* no downward motion of the active weight, results in:

$$P\alpha + Q\beta + R\gamma + \dots = 0 \quad (5.1)$$

“which is precisely the analytic expression of the general principle of virtual velocities”.

Whatever its ingenuity, this proof obviously still suffers some shortcomings which do not appear in the proof given by FOURIER in the same issue of the *Journal de l'école polytechnique*²⁷. Anyhow, the equation (5.1) is the starting point of the most important development in the subsequent editions of the *Mécanique analytique* regarding geometrical constraints.

5.4 LAGRANGE'S MULTIPLIERS

Substituting differential quantities for $\alpha, \beta, \gamma, \dots$, may be written in the general differential form with respect to the coordinates (x_i, y_i, z_i) of each body (index i) of the system:

$$P dp + Q dq + R dr + \dots = 0 \quad (5.2)$$

where the differentials dp, dq, dr, \dots are typically written

$$dp = \frac{\partial p}{\partial x_i} dx_i + \frac{\partial p}{\partial y_i} dy_i + \frac{\partial p}{\partial z_i} dz_i \quad (5.3)$$

with the index i referring to the body concerned by the considered force. In the case of no geometrical constraints the equation (5.2) is valid $\forall dx_i, dy_i, dz_i, \forall i$. This leads to the equilibrium equations of the system by equating all the coefficients of the $dx_i, dy_i, dz_i, \forall i$ to zero.

Assuming the geometrical constraints that may be imposed to the evolution of the bodies to be written as linear forms of the differentials $dx_i, dy_i, dz_i, \forall i$ being equal to zero:

$$dL = 0, \quad dM = 0, \quad dN = 0, \dots \quad (5.4)$$

²⁷ (Fourier, 1797).

LAGRANGE remarks, from the theory of linear equations, that writing (5.2) with (5.3) under the mathematical constraints (5.4) on $dx_i, dy_i, dz_i, \forall i$ is equivalent to writing

$$Pdp + Qdq + Rdr + \dots + \lambda dL + \mu dM + \nu dN + \dots = 0, \forall dx_i, dy_i, dz_i, \forall i \quad (5.5)$$

where λ, μ, ν are indeterminate.

To this method LAGRANGE gave the name of *Méthode des multiplicateurs* (Multiplier Method) while referring to (5.5) as the general equation of equilibrium and explaining how to handle it in order to obtain the solution to the equilibrium problem.

But the most important step forward came out from his noting the mathematical similarity of the $\lambda dL, \mu dM \dots$ terms with the $Pdp, Qdq \dots$ ones and giving a mechanical significance to the Lagrange multipliers. Assuming the linear form dL to be the differential of a function L of the coordinates of the bodies in the system, the term λdL is written

$$\lambda dL = \lambda \frac{\partial L}{\partial x_i} dx_i + \lambda \frac{\partial L}{\partial y_i} dy_i + \lambda \frac{\partial L}{\partial z_i} dz_i, i = 1, 2, \dots, \quad (5.6)$$

which is quite similar to (5.3) but for the fact that the coordinates of more than one body may be involved. Thence LAGRANGE's statements:

“It comes out then that each geometrical constraint equation is equivalent to one or several forces acting on the system, along given directions or, as a general rule, tending to vary the values of the given functions; so that the same state of equilibrium will be obtained for the system, either using these forces or the constraint equations. And here one encounters the metaphysical reason why introducing the terms $\lambda dL + \mu dM + \dots$ in the general equilibrium equation makes it possible to treat this equation as if all bodies were completely free.”

and further on:

“Conversely, these forces may be substituted for the geometrical constraint equations in such a way that, using these forces the constituent bodies of the system will be considered as completely free without any constraint... In proper words, these forces stand as the resistances that the bodies should meet for being linked to each other or due to the obstacles that may impede their motion; or rather, these forces are precisely the resistances, which must be equal and opposite to the pressures exerted by the bodies.”

The scalars λ, μ, ν are now known as the *Lagrange multipliers* associated with the corresponding constraints.

These statements are crucial: one may say that they introduce and define binding and internal forces from the given geometrical constraints, either external or internal, through the concept of duality. Compared with the initial definition of forces by LAGRANGE (§5.1), we observe that these forces are *defined through the movement they are supposed to impede*. It follows that, practically, resistances will not have a data status but be characterised by a limitation imposed on their magnitude.

This actually opens the way to the formulation of the principle that is used now when modelling mechanical systems. Without getting into too many details, we must mention the contribution by FOSSOMBRONI²⁸ and the statement by FOURIER²⁹:

”Moreover, if one regards resistances as forces, which provides, as it is known, the means of estimating these resistances, the body can be considered free, and sum of the moments is nil for all possible displacements.”

One may also remark that although LAGRANGE’s proof assumes the geometrical constraints, either external or internal, to be written as linear forms of $dx_i, dy_i, dz_i, \forall i$ being equated to zero, the final interpretation of the scalars λ, μ, ν , related to the resistances associated with these geometrical constraints, yields the possibility of treating geometrical constraints that are expressed as inequalities such as $dL \geq 0$ and/or $dM \geq 0$ (as for unilateral support for instance): the geometrical constraints are still considered as equalities, which we may call “bilateral”, $dL = 0, dM = 0$, and inequalities $\lambda \leq 0, \mu \leq 0$ are imposed on λ, μ as conditions on the “resistances”. This maintains the essential point that the equation (5.5) is written $\forall dx_i, dy_i, dz_i, \forall i$, as remarked by FOURIER.

6 STATICS AND DYNAMICS

As this contribution is devoted to Statics no mention has been made up to now of NEWTON’s³⁰ statements in the *Principia* (Newton, 1687) in the form of the two first laws that express what is usually named as the *law of inertia* (after GALILEO’s) and *fundamental law of dynamics*.

Galilean reference frames are defined for the validity of the fundamental law where the external force exerted on a body is proportional to its acceleration:

$$\text{in a Galilean reference frame } \underline{F} = m\underline{a}. \quad (6.1)$$

²⁸ (Fossombroni, 1796) which FOURIER acknowledges in (Fourier, 1798), *Œuvres publiées...*, p. 518.

²⁹ (Fourier, 1798), *Œuvres publiées...*, p. 488: ”Au reste, si l’on considère les résistances comme des forces, ce qui fournit, comme on le sait, le moyen d’estimer ces résistances, le corps peut être regardé comme libre, et la somme des moments est nulle pour tous les déplacements possibles.”

³⁰ Isaac NEWTON (1642-1727).

D'ALEMBERT's³¹ principle (d'Alembert, 1743) transforms the equation (6.1) into

$$\text{in a Galilean reference frame } \underline{F} - m\underline{a} = 0 \quad (6.2)$$

where $(-m\underline{a})$ plays the role of a fictitious external force, the *inertia force*, that equilibrates the genuine external force \underline{F} . Thus, when studying a system, adding the inertia forces to the external forces exerted on each element makes it possible to deal with Dynamics in the same way as Statics through the principle of virtual velocities.

7 REFERENCES AND BIBLIOGRAPHY

ALEMBERT, J. (LE ROND D'), (1743) – *Traité de dynamique*. David l'aîné, Paris.

ARCHIMEDES – *De Planorum Aequilibriis*. (A treatise on the equilibrium of planes or their centres of gravity).

ARISTOTLE – *Physicae Auscultationes*. (Lectures on nature). Translated by R. P. Hardie and R. K. Gaye. <http://classics.mit.edu/Aristotle/physics.7.vii.html>.

ARISTOTLE (long attributed to) – *Quaestiones Mechanicae*. (Mechanical Problems). http://penelope.uchicago.edu/Thayer/E/Roman/Texts/Aristotle/Mechanica*.html

BENVENUTO, E., (1981) – *La Scienza delle Costruzioni e il suo Sviluppo Storico*. Sansoni, Florence.

BENVENUTO, E., (1991) – *An Introduction to the History of Structural Mechanics*. Part I, *Statics and Resistance of Solids*. Springer Verlag, New York.

CAPECCHI, D., (2012) – *History of Virtual Work Laws : A History of Mechanics Perspective*. Springer-Verlag Italia.

DE GROOT, J., (2008) – Dunamis and the Science of Mechanics: Aristotle on Animal Motion. *Journal of the History of Philosophy*, vol. 46, 1,43-68.

DESCARTES, R., (1668) – *Epistolae*, Amsterdam, 1668. *Correspondance.1* (April 1622-February 1638); *Correspondance.2* (March 1638-December 1639); *Correspondance.4* (July 1643-April 1647). Published by Adam, C. and Tannery, P., 1969. Librairie philosophique J. Vrin, Paris, 1988.

DUGAS, R., (1950) – *Histoire de la Mécanique*. Éditions du Griffon, Neuchatel.

DUHEM, P. M. M., (1905) – *Les origines de la Statique* vol. 1, Hermann, Paris.

DUHEM, P. M. M., (1906) – *Les origines de la Statique* vol. 2, Hermann, Paris.

FOSSOMBRONI, V., (1796) – *Memoria sul principio delle velocità virtuali*, Florence.

FOURIER, J., (1797) – Mémoire sur la statique contenant la démonstration du principe des Vitesses virtuelles, et la théorie des Momens. *Journal de l'école polytechnique*. V^e cahier, Tome II, prairial an VI, pp. 20-60, Imprimerie de la République, Paris. *Œuvres de Fourier publiées par les soins de M. Gaston Darboux*, II, Gauthier-Villars, Paris, 1890.

³¹ Jean LE ROND D'ALEMBERT (1717-1783).

GALILEO GALILEI (1599) – *Le Meccaniche. Della vite*.
http://it.wikisource.org/wiki/Le_mecaniche/Della_vite

GALILEO GALILEI (1634) – *Les Mécaniques*. Translation by Mersenne of *Della Scienza Meccanica*, Ravenna (1649).

GALILEO GALILEI (1638) – *Discorsi e dimostrazioni matematiche intorno a duo nuove scienze attenenti alla Meccanica, ed ai movimenti locali*. Elsevirii, Leyden. *Dialogues concerning two new sciences*. Crew, H. & Salvio, A. transl., Dover, New York, 1954.

LAGRANGE, J-L, (1788) – *Mécanique Analytique*. À Paris, chez la Veuve Desaint, rue du Foin S. Jacques.

LAGRANGE, J-L, (1888) – *Œuvres complètes*. Vol. 11. Gauthier-Villars, Paris, 1888.

LAGRANGE, J-L. (1797) – Sur le principe des vitesses virtuelles. *Journal de l'école polytechnique*. V^e cahier, Tome II, prairial an VI, pp. 115-118, Imprimerie de la République, Paris.

LEONARDO DA VINCI (1487-1508) – *Les Manuscrits de Léonard de Vinci*. Ms A-M. Bibliothèque de l'Institut de France, Paris.

NEWTON, I., (1687) – *Philosophiæ naturalis principia mathematica*.
<http://www.gutenberg.org/files/28233/28233-pdf.pdf>.

The Principia, translation by Andrew Motte, 1729;
<http://www.archive.org/stream/newtonspmathema00newtrich#page/n7/mode/2up>.

The Principia, A new translation by I.B. Cohen and A. Whitman, University of California press, Berkeley, 1999.

STEVIN, S. (1586/1605) – *Beghinselen der Weeghconst*. Leyden. *The principal works of Simon Stevin*, vol. I, general introduction, Mechanics, edited by E.J. Disksterhuis, Amsterdam, C. V. Swets & Zeitlinger, 1955.

STEVIN, S. (1605/1608) – *Hypomnemata mathematica. (Additamenti Staticæ pars secunda: de Trochleostatica)*. Lugodini Batavorum, ex officina Ioannis Patii, academiæ Typographi.

VARIGNON, P. (1725) – *Nouvelle Mécanique ou Statique*, vol. II. Claude JOMBERT, Paris.

WALLIS, J., (1670) – *Mechanica: sive, De Motu, Tractatus Geometricus. Pars Prima*.