From Galileo to Convexity: Some Key Ideas in Structural Mechanics

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Abstract

The main topic treated in the *First Day* and the *Second Day* of Galileo's *Dialogues* is the resistance that solids offer to fracture with special consideration to prisms and cylinders submitted to axial tensile loading or to "transverse", *i. e.* bending, forces. Although no consideration is given to deformation of the solid before fracture one may say that Galileo implicitly introduces the concept of a Continuum within which coherence forces do act in order to maintain the filaments, fibres or any other constituent particles together. Thus, he opens the way to the concept of stress, which was settled explicitly some 200 years later. Having recognised that coherence forces and gravity forces in a solid are not related in the same way to its geometric scale, he performs what can be considered as the first striking example of dimensional analysis with application to similarity. In its celebrated analysis of the resistance of a cantilever beam submitted to bending, Galileo gives a first attempt to deriving the resistance of a whole solid submitted to some kind of loading from the resistance of its constituent material determined from another test. Together with Coulomb's celebrated *Essay* these are two milestones of the Theory of yield design, the fundamental root of *Ultimate Limit State Design*, which is presently introduced in international codes for civil engineering.

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THE DIALOGUES

According to his writings to his faithful friend Elia Diodati, Galileo considered the New Sciences as "*superior to everything else of his hitherto published".* Although the foundations of this work were laid as early as during Galileo's stay in Padua, it was given its final form during Galileo's five month enforced residence in Siena after his trial and condemnation: he then writes that he has completed *"a treatise on a new branch of mechanics full of interesting and useful ideas".* Publication of the work in Italy proved impossible with the answer from the Inquisitor that there was an express order prohibiting the printing or reprinting of any work of Galileo, either in Venice or in any other place, *nullo excepto*. This is the reason why this illustrious book appeared in Leyden after Louis Elzevir had arrived in Italy in 1636 and taken the manuscript with him on his return home[1].

The goal of this paper is to show how some fundamental concepts in Structural and Continuum Mechanics, which were developed during the following centuries and found practical applications in the design of structures, were settled by Galileo in this famous work. For this purpose, excerpts of this book will be quoted from the English translation of the

[1] This short introduction is derived from the preface written by Professor Antonio Favaro of the University of Padua to the 1st edition of the English translation [2] published in 1914.

Discorsi e Dimostrazioni Matematiche intorno à due nuove scienze [1] by Henry Crew and Alfonso de Salvio [2].


DISCORSI
E
DIMOSTRAZIONI
MATEMATICHE,
intorno à due nuoue scienze
Attenenti alla
MECANICA & i MOVIMENTI LOCALI;
del Signor
GALILEO GALILEI LINCEO,
Filosofo e Matematico primario del Serenissimo
Grand Duca di Toscana.
Con vna Appendice del centro di grauità d'alcuni Solidi.
IN LEIDA,
Appresso gli Elsevirii. M. D. C. XXXVIII.

The paper will obviously just provide a mere idea and simple examples of the “Key Ideas” announced in the title.

DIMENSIONAL ANALYSIS

The *First Day* begins by considerations that we would now qualify as dimensional analysis. Indeed, attention is given by the dialogists to the possibility or impossibility of increasing proportionally the size of a structure or a *“machine*”. It is just asking the question whether it is possible to extrapolate a real size structure from a reduced scale one.

> *“Yet I shall say and will affirm that, even if the imperfections did not exist and matter were absolutely perfect, unalterable and free from all accidental variations, still the mere fact that it is matter makes the larger machine, built of the same material and in the same proportion as the smaller, correspond with exactness to the smaller in every respect except that it will not be so strong or so resistant against violent treatment; the larger the machine, the greater its weakness.”*

As a matter of fact, in this statement the so-called *“violent treatment”* is implicitly just related to the action of gravity forces in the structure or in the machine. Examples are given such as:

> *“If we take a wooden rod of a certain length and size, fitted, say, into a wall at right angles, i. e., parallel to the horizon, it may be reduced to such a length that it will just support itself; so that if a hair’s breadth be added to its length it will break under its own weight and will be the only rod of the kind in the world.”*

Without any calculation, the competition between the action of gravity forces and some kind of resistance forces in the volume of the rod, which will be examined later on in the same *Day*, is already underlying here. At the present stage, the statement that *“the larger the machine, the greater its weakness”* leads us to guessing that the two kinds of forces that are in competition are related in a different way to the scale of the solid under consideration.

Jumping to the *Second Day*, after acknowledging the fact that resisting forces will refer to surfaces while gravity forces obviously refer to volumes the conclusion comes:

> *"From what has already been demonstrated, you can plainly see the impossibility of increasing the size of structures to vast dimensions either in art or in nature; ... ; nor can nature produce trees of extraordinary size because the branches would break down under their own weight; so it would be impossible to build up the bony structures of men, horses, or animals so as to hold together and perform their normal functions if these animals were to be increased enormously in height; for this increase in height can be accomplished only by employing a material which is harder and stronger than usual, or by enlarging the size of the bones, thus changing their shape until the form and appearance of the animals suggest a monstrosity."*

Figure 2 shows the famous drawing in which Galileo has "*Sketched a bone whose natural length has been increased three times and whose thickness has been multiplied until, for a correspondingly large animal, it would perform the same function which the small bone performs for its small animal."*

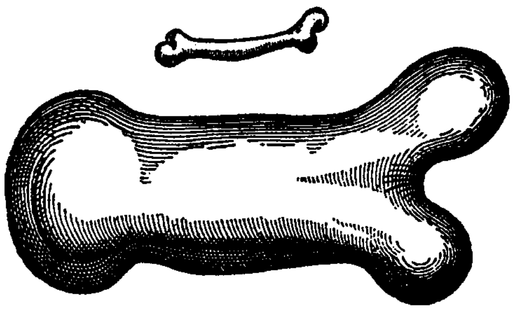


Fig. 27

Figure 1. Figure 27 in Galileo's *Dialogues*.

Although they may not be aware of its existence this analysis by Galileo anticipates the current practise of civil engineers who commonly use reduced scale experiments as one of their many approaches when designing a structure, especially when this structure does not, more or less, reproduce any existing one, due to its dimensions or to the specific actions and special constraints that might be imposed to it. When designing the foundations of a bridge for instance different materials are involved at the same time. For some of them, whose constitutive law is well characterised and simple enough, it is possible to have different materials corresponding to each other in the full-scale structure and in the reduced scale model in a proper way. But civil engineers also need to cope with soil and similar materials on which their structures are to be safely built and remain stable. The behaviour of these materials is complex and dependent on the level of internal efforts they are submitted to. It is therefore almost compulsory for reduced scale experiments to be representative to keep the material unchanged and to substitute the gravity forces with body forces artificially increased in inverse proportion with respect to the scale of the model: this is the reason why the technique of centrifuge modelling is commonly used in such cases as well as others such as the hydraulic gradient. Figure 2 shows a centrifuge experiment performed for the design of the foundations of the Rion-Antirion Bridge in Greece.

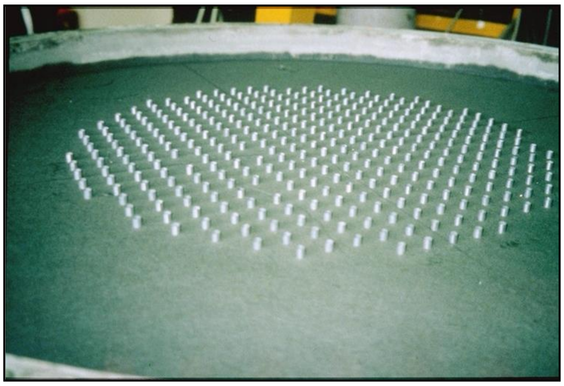


Figure 2. Centrifuge test of a reduced scale model of a reinforced soil foundation for the Rion-Antirion Bridge in Greece.

Presently, dimensional analysis currently refers to the so-called Π-theorem, which is usually attributed to Buckingham [3] although it had been established by Vaschy [4] some 20 years earlier. In its most popular form this theorem leads to a practical method for exhibiting the non-dimensional factors that govern a physical phenomenon. It thus makes the resolution of the problem shorter, reducing the amount of numerical calculations to be performed, and the presentation and analysis of the results easier. It also gives clear similarity rules for representative reduced scale experiments.

Non-dimensional factors, such as the famous Reynolds number, Mach number, etc., are so popular among Fluid mechanicists that students in engineering sometimes forget that they are relevant as well in structural Mechanics. Moreover, it turns out that the same students, since they are using the International System of units (SI), often do not even care about the physical nature of the quantities they are handling and obviously not about dimensional analysis. It is no paradox that Galileo's dimensional reasoning is so similar to considerations developed by Nobel Prize winner Pierre-Gilles de Gennes and his co-workers for instance: just pure Physics.

As a matter of fact the mathematical bases of dimensional analysis are not at all trivial. The original idea lies in the feeling that any physical relationship should be independent of the units the observer chooses for each quantity involved in its writing. It follows that it should be possible to have a continuous variation of the units without any change occurring in the relationship. Any problem being set through a system of field differential equations with boundary conditions, depending on the time, where given values of some quantities or fields are the data, the solution to the problem is hopefully a general relationship where the unknown quantities or unknown fields are determined as functions of those data. Proving that this relation only involves non-dimensional factors built with all quantities appearing in the problem is a hard mathematical work for which one often refers to a theorem by Federman [5]. But it is only in more recent papers [6] that, making use of some results in topology related to properties of arc-wise connected subgroups of $\mathbb{R}^n$ [7,8,9] that a full mathematical proof of Vaschy-Buckingham theorem has been produced.

THE CONTINUUM AND ITS COHERENCE

Coming back to the *First Day* in the *Dialogues*, we will now see how Galileo meets the concept of a Continuum together with "*coherence*" of matter:

> *"To grasp this more clearly, imagine a cylinder or a prism, AB, made of wood or other solid coherent material. Fasten the upper end, A, so that the cylinder hangs vertically. To the lower end, B, attach the weight C. It is clear that however great they may be, the tenacity* [tenacità] *and coherence* [coerenza] *between the parts of this solid, so long as they are not infinite, can be overcome by the pull of the weight C, a weight which can be increased indefinitely until finally the solid breaks like a rope. And as in the case of the rope whose strength we know to be derived from a multitude of hemp threads which compose it, so in the case of the wood, we observe its fibres and filaments run lengthwise and render it much stronger than a hemp rope of the same thickness. But in the case of a stone or metallic cylinder where the coherence seems to be still greater the cement which holds the parts together must be something other than filaments and fibres; and yet even this can be broken by a strong pull."*

As appears here Galileo is aware that the resistance of the piece of solid coherent material he is testing comes from the solidarity between its constituent elements. Since he is not concerned with the evolution of the deformation of this solid when the load is increased he only imagine internal forces, which hold the parts together, as resisting to fracture. Without undue extrapolation it may also be guessed that those internal forces, since they hold parts together, are implicitly contact forces between the elements and, since homogeneity is implicitly assumed, that they are uniformly distributed over the area of the cross section.

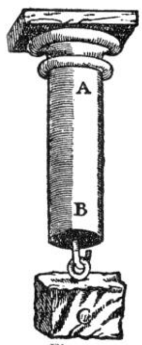


Fig. 1

Figure 3. Figure 1 in Galileo's *Dialogues*.

In the *Second Day*, with the so often referred to problem of the cantilever beam, this concept of resisting internal forces is made clearer. The purpose of the analysis is to explain why the resistances of a beam to axial forces (tension) and to transverse forces (bending) are so different. As appears in the excerpt given hereunder, the resistance of the beam in tension is located and uniformly distributed in its cross section. This may be taken as a first step towards the concept of a stress vector, which in this case is normal to the considered cross section.

Let us imagine a solid prism ABCD fastened into a wall at the end AB, and supporting a weight E at the other end; understand also that the wall is vertical and that the prism or cylinder is fastened at right angles to the wall. It is clear that, if the cylinder breaks, fracture will occur at the point B where the edge of the mortise acts as a fulcrum for the lever BC, to which the force is applied; the thickness of the solid BA is the other arm of the lever along which is located the resistance. This resistance opposes the separation of the part BD, lying outside the wall, from the portion lying inside.

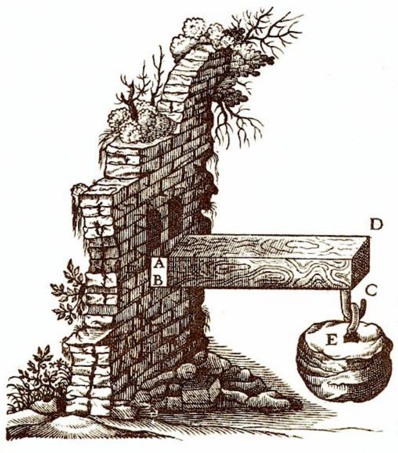


Figure 4. Figure 17 in Galileo's *Dialogues*.

Since this paper does not pretend to an exhaustive survey let us jump to Coulomb's *Essay* [10]. Internal forces are explicitly introduced through resisting forces together with the concepts of friction and cohesion. Taking as an example the stability analysis of a retaining wall treated as a 2-dimensional problem, a potential failure line *Beg* is drawn within the bulk of soil and the equilibrium of the mass of soil *CBeg* is examined under the action of gravity forces, of the thrust of the wall and taking into account the resistance to rupture along *Beg*. This resistance is exerted through normal and tangential forces distributed along the line.

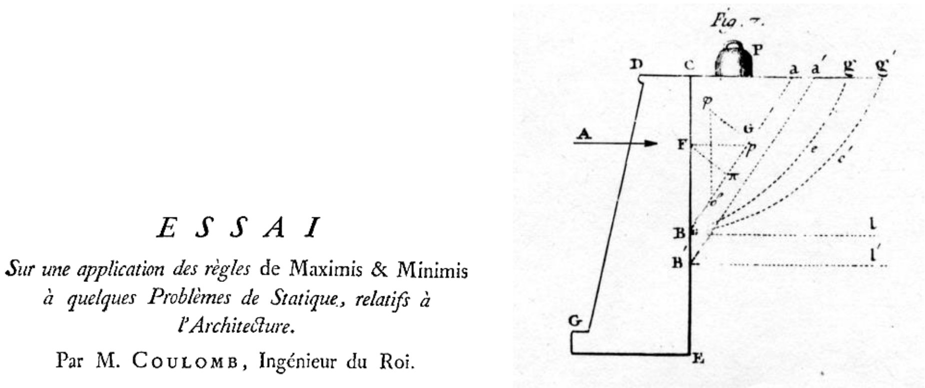
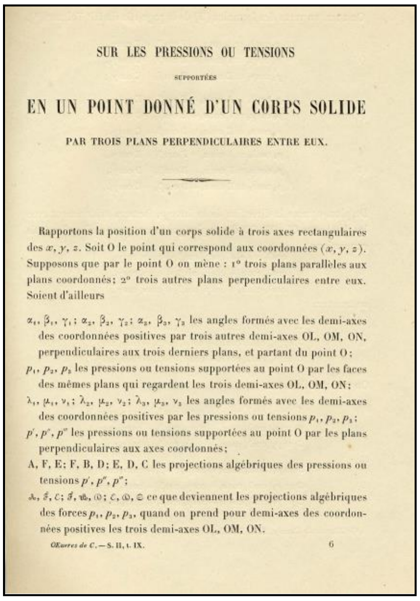


Figure 5. Stability analysis of a retaining wall from Coulomb's *Essay*.

As a short cut we may say that, although it is only introduced as resisting forces in a stability analysis as in Galileo's work, the concept of a stress vector (as we would call it now) is present at that stage: on any surface element along line *Beg* the actions of the whole bulk of soil on the right-hand side are substituted by an elementary force.

The concept is plainly settled by Cauchy [11] without any reference to stability analysis. At any point within a solid body he introduces the 9 components of the surface density of forces exerted on three mutually orthogonal plane elements by the neighbouring constituent particles. Through equilibrium he derives that on any other plane element at the same point the components of the corresponding surface density are determined from the 9 preceding ones and depend linearly on the orientation of the plane. Finally he establishes the symmetry of the 9-component array and the equations of dynamics.



SUR LES PRESSIONS OU TENSIONS

SUPPORTÉES

EN UN POINT DONNÉ D'UN CORPS SOLIDE

PAR TROIS PLANS PERPENDICULAIRES ENTRE EUX.

Rapportons la position d'un corps solide à trois axes rectangulaires des x, y, z. Soit O le point qui correspond aux coordonnées (x, y, z). Supposons que par le point O on mène : 1° trois plans parallèles aux plans coordonnés; 2° trois autres plans perpendiculaires entre eux. Soient d'ailleurs

$\alpha_1, \beta_1, \gamma_1; \alpha_2, \beta_2, \gamma_2; \alpha_3, \beta_3, \gamma_3$ les angles formés avec les demi-axes des coordonnées positives par trois autres demi-axes OL, OM, ON, perpendiculaires aux trois derniers plans, et partant du point O;

p_1, p_2, p_3 les pressions ou tensions supportées au point O par les faces des mêmes plans qui regardent les trois demi-axes OL, OM, ON;

$\lambda_1, \mu_1, \nu_1; \lambda_2, \mu_2, \nu_2; \lambda_3, \mu_3, \nu_3$ les angles formés avec les demi-axes des coordonnées positives par les pressions ou tensions p_1, p_2, p_3;

p', p'', p''' les pressions ou tensions supportées au point O par les plans perpendiculaires aux axes coordonnés;

A, F, E; F, B, D; E, D, C les projections algébriques des pressions ou tensions p', p'', p''';

$\mathcal{A}, \mathcal{F}, \mathcal{E}; \mathcal{F}, \mathcal{B}, \mathcal{D}; \mathcal{E}, \mathcal{D}, \mathcal{C}$ ce que deviennent les projections algébriques des forces p_1, p_2, p_3, quand on prend pour demi-axes des coordonnées positives les trois demi-axes OL, OM, ON.

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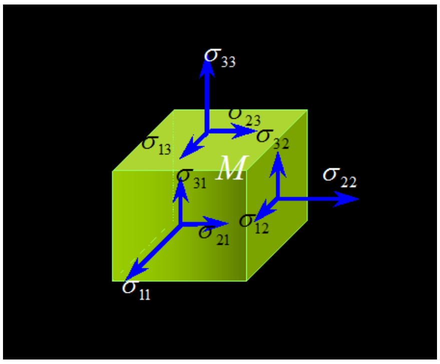

Figure 6. Cauchy's Memoir *On pressures and tensions acting on three mutually orthogonal planes at a given point of a solid body*. Presentation of the 9 components of the surface density of forces.

This classical presentation of the Cauchy stress tensor appears in most textbooks in Continuum Mechanics because it offers a concrete vision of the corresponding mathematical concept. It is also remarkable that, although Cauchy's memoir was devoted to *solid* body, the versatility of the concepts of stress vector and stress tensor is such that they can be applied to any material ranging from solids to fluids, provided the interactions between the constituent particles may be modelled through a surface density of contact *forces*.

Going back to Galileo and Coulomb, we have seen that internal forces were in each case associated with a potential or virtual motion considered for the collapse in the bearing capacity problem (rotation around the fulcrum *B* in Figure 4) or in the stability analysis problem (rupture along *Beg* in Figure 5). This reveals the mathematical duality between internal forces and deformation of matter, even if, in the cases under concern, only rupture is considered. The virtual work method, which is sometimes adopted for modelling external and internal efforts in Continuum Mechanics [12], follows that track. It offers a general mathematical framework that can be applied safely for modelling external and internal forces in a system once its geometrical modelling and its evolution with respect to time have been described, starting from physical considerations.

As already noticed, both Galileo's and Coulomb's analyses rest on the same reasoning. Generally speaking the question to be answered is that of the resistance of a whole body submitted to some kind of loading when the resistance of its constituent material is given.

In figure 4, we may say that the material is characterised by the resistance of the fibres in tension as derived from the test in Figure 3; Galileo being concerned in explaining the difference between the difference of resistance of the beam to tension and to bending he does not speak of any resistance in compression, which amounts to assuming it is infinite. Considering that the cantilever beam acts as a lever with *B* as its fulcrum results in writing the balance between the moments at point *B* of the active load at point *C* and of the resistance forces distributed in the cross section *BA*, as sketched in Figure 7. It yields a maximum value for the weight *E*.

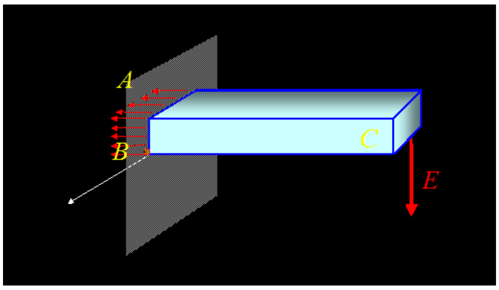

Figure 7. The principle of Galileo's reasoning.

Galileo's reasoning for this problem has often been criticised due to the fact that equilibrium for the horizontal forces is apparently not satisfied. As a matter of fact it turns out that, provided the resistance of the fibres in compression is assumed to be infinite, it would only require one horizontal fibre in *B* to be in compression to fulfil the condition without changing the result for the maximum value of *E*. Nevertheless other criticisms can be made and an interpretation of Galileo's solution will be given later on.

The analysis performed by Coulomb is by far more sophisticated. Again the balance between active forces and resisting forces is the corner stone of the reasoning. But resisting forces are no more one-dimensional and the resistance of the material is defined through a condition imposed on the normal and tangential components of the density of the resisting forces acting along *Beg*. This means that a domain of resistance is prescribed for the material through the famous Coulomb's criterion. Also, Coulomb is aware that his partition of the bulk of soil is just arbitrary and that he must look for the one whose equilibrium is most unfavourable although he performs the calculation for straight lines *Ba* only.

The common features of these two analyses are the basic principles of the Theory of Yield Design [13] in its most simple form. The specific ways in which they were performed are just practical examples of two general methods for implementing this theory.

The data of a yield design problem are:

- The geometry of the system, on which all the analysis will be performed without taking any change into account;
- The loading process of the system, which usually depends on a few number of scalar parameters;

- The resistance of the constituent material, which is characterised, as in Coulomb's analysis, through a domain of resistance prescribed, at each point of the system, to the stress state. This domain is convex.

The question to be answered from those data is to determine the set of loads that can be sustained by the system taking into account the resistance of its constituent material.

Obviously, since the complete behaviour of the constituent material is not described in the data, no definitive answer to that question should be expected. It only means that the conclusions derived from the theory will be checked experimentally, which is just common sense.

Nevertheless by just referring to the requirement that equilibrium equations should be satisfied all over the system at the same time as the resistance of the material should not be exceeded we are able to determine the domain of potentially safe loads, meaning that the system cannot sustain any load that stands outside this domain. Convexity is preserved in passing from the material to the system, which means that the domain of potentially safe loads is a convex in the vector space of the loads. This property makes its determination easier.

A first method starts from the definition and provides interior estimates of the domain through maximization procedures and convex combinations of potentially safe loads from the construction of statically admissible and safe stress fields (Figure 8).

A second method takes advantage of the duality between internal forces (the stress tensor) and the virtual strain rates of the material. Through the principle of virtual work the domain of resistance of the constituent material is defined, at any point of the body, through its support function whose argument is the virtual strain rate. This function is the volume density of maximum resisting rate of work [2]. The method then requires the construction of virtual velocity fields or virtual collapse mechanisms, which happens to be usually easier than the construction of statically admissible and safe stress fields. A necessary condition for safe loads states that in any such mechanism the virtual rate of work of the external forces must remain inferior to the maximum resisting rate of work. Through minimization procedures this method provides exterior estimates for the domain (Figure 8).

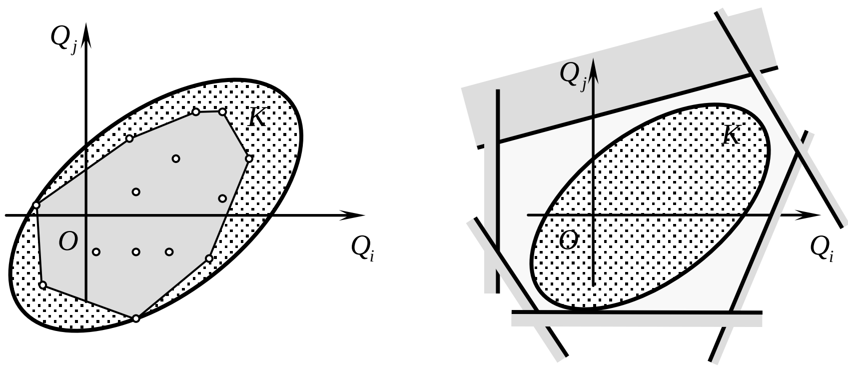

Figure 8. Interior and exterior estimates for the domain of potentially safe loads.

Galileo's reasoning fits in this framework as an implementation of the exterior estimate method where a rotation about fulcrum *B* is used as a virtual collapse mechanism. For this reason the result so obtained remains valid as an upper bound to the weight *E* whatever the fibre resistance in compression.

[2] This point of view was considered by Prager [14] in the theory of Plasticity.

Coulomb's analysis is also an implementation of the exterior estimate method although the underlying virtual velocity field does not come out straightforwardly. Restricting ourselves to the case of straight lines *Ba*, which corresponds to the famous Coulomb's wedge, if the considered soil exhibits friction the virtual collapse mechanism associated with Coulomb's analysis is a rigid body translation of the volume *BCa* with both sliding and parting from the bulk of soil along *Ba*. This virtual mechanism is simply determined from the mathematical rules of duality applied to Coulomb's criterion.

Due to its general mathematical formulation the theory of yield design covers many methods, which have been used for long by civil engineers for the design of vaults, plates, shells, retaining walls, slope stability analysis, etc. Also, clear reference to duality for virtual collapse mechanisms overcomes the shortcomings encountered with some of these methods in their classical forms based upon intuitive considerations. Moreover, it makes it possible to combine these methods when necessary for the analysis of a whole structure.

New design codes are being presently proposed in Europe in civil engineering, which include *Ultimate Limit State Design* as one approach for assessing the safety of structures. Roughly speaking this approach can be expressed by the following symbolic equation:

$$R_{\mathrm{d}} \geq S_{\mathrm{d}}$$

which must be satisfied at equilibrium, where S_{d} stands for the "effect" of the *design* loads and R_{d} stands for the "effect" of the *design* resistances. Obviously this equation, which meets any common feeling about safety, must be turned into a quantitative form for practical design applications. The theory of yield design provides a theoretical basis for this approach for many reasons:

- As Coulomb himself did in its *Memoir*, it makes a clear distinction between loads and resistances: loads are prescribed while resistances are just bounds that must not be exceeded. This helps defining the design values of these quantities properly through partial safety coefficients that are given in the codes.
- It relies only on the requirement that equilibrium under the prescribed loads should be possible without exceeding the material resistances, which meets exactly the terms of reference of *Ultimate Limit State Design*.
- It is able to handle multi-parameter loading processes, as it is always the case for structures in practice. The property of convexity, which has been established in the theory, gives the proof that only extreme values of the design loads should be combined using properly defined partial safety coefficients.
- It makes it possible to give a physical and quantitative meaning to the "effects" that appear in the equation, whatever the complexity of the structure. As an example, addressing Galileo's analysis of the cantilever beam within the framework of *Ultimate Limit State Design*, we see that the "effects" he considers are the moments about fulcrum *B*, a choice which is questionable. The theory of yield design would define the "effects" R_{d} and S_{d}, starting from the exterior estimate method, through a procedure that amounts to looking for the most unfavourable case for the structure. Depending on the way this procedure is implemented in practice, it leads to the introduction of a (method) global safety factor.

This token of topicality of Galileo's approach brings this paper to its conclusion.

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