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Ultimate bearing capacity of shallow foundations under inclined and eccentric loads. Part II: purely cohesive soil without tensile strength

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ABSTRACT. – The problem of determining the bearing capacity of a strip footing resting on the surface of a homogeneous half space and subjected to an inclined, eccentric load, is solved within the framework of the yield design theory assuming that the soil is purely cohesive without tensile strength according to Tresca's strength criterion with a tension cut-off. The soil foundation interface is also purely cohesive, in terms of the homologous strength criterion with a tension cut-off.

As in a companion paper [Salençon & Pecker, 1995], both the static and the kinematic approaches of the yield design theory are used. New stress fields are constructed, in order to comply with the condition of a tension cut-off within the soil medium, and new lower bounds are determined as substitutes to those given in the companion paper. Velocity fields taking advantage of the tension cut-off contribution in the expression of the maximum resisting work are also implemented, giving new lower bounds.

As may be expected from common sense and from the general results of the theory, it appears that the tension cut-off condition within the soil medium results in lower values of the bearing capacity of the foundation, and that the gravity forces acting in the soil mass have a stabilizing effect.

1. Introduction

The problem solved in the current study, relates to the ultimate bearing capacity of a strip footing resting on the surface of a homogeneous half space and subjected to an inclined, eccentric load. The foundation, with width B, is assumed to be rigid and to have an infinite length. It is subjected to uniformly distributed external loads along the direction Z (*Fig.* 1). The evaluation of the ultimate bearing capacity is obtained within the framework of the yield design theory [Salençon, 1983, 1993], as detailed in the companion paper [Salençon & Pecker, 1995].

In the companion paper, the problem is studied assuming the soil to be purely cohesive, in accordance with Tresca's strength condition, while the interface between the soil and the foundation is also purely cohesive, with a similar cohesion and no normal tensile strength. In the present case, a tension cut-off condition is added to the soil strength criterion, describing a purely cohesive medium with no tensile strength. As in the companion paper, other interface strength conditions can be easily taken into account.

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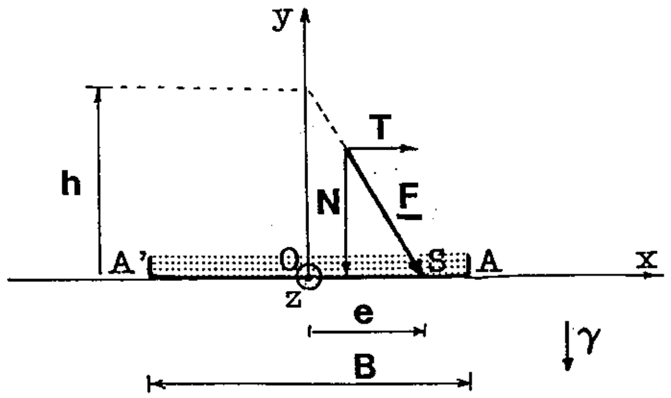

Fig. 1. – Strip foundation under inclined, eccentric load.

Under aforementioned assumptions, the problem can be studied within the framework of the plane strain yield design theory, as defined in [Salençon, 1983, 1993].

Numerous publications (*i.e.* [Meyerhof, 1953-1963; Hansen, 1961-1970; Tran Vo Nhiem, 1971; Khosravi, 1983; Swami Saran & Argawal, 1991]) deal with the problem of load inclination and load eccentricity for a purely cohesive soil; however, to the best of the authors' knowledge, the problem for a cohesive soil without tensile strength has never been solved before.

2. Theoretical framework

The notations, identical to those of the companion paper, are recalled in Figure 1. We denote γ as the unit weight of the soil:

(1) $$\underline{\gamma} = -\gamma\, \underline{e}_y$$

and

(2) $$\underline{F} = -N\, \underline{e}_y + T\, \underline{e}_x, \qquad \underline{M} = -M\, \underline{e}_z$$

are the force system resultants, computed at O, the mid-point of the foundation $A'\,A$; $\underline{e}_x$, $\underline{e}_y$ and $\underline{e}_z$ are the unit vectors of the cartesian coordinate axes Ox, Oy, Oz.

Let S be the point of application of $\underline{F}$ on Ox and e the algebraic eccentricity which is positive along Ox; it follows that:

(3) $$M = N\, e, \qquad |e| \le B/2$$

For pratical situations, the load eccentricity onto the foundation may arise due to an elevated point of application of the horizontal force component $\underline{T} = T\, \underline{e}_x$; hence:

(4) $$M = T\, h, \qquad h \ge 0$$

The soil is assumed to be purely cohesive without tensile strength. The strength criterion using Tresca's criterion with tension cut-off is:

(5) $$f\,(\underline{\underline{\sigma}}) = \operatorname{Sup}\,(|\sigma_1 - \sigma_2| - 2\,C,\ \sigma_1,\ \sigma_2) \le 0$$

where σ_1 and σ_2 are the in-plane principal stresses of the stress tensor $\underline{\underline{\sigma}}$; C designates the soil shear strength and tensile stresses are positive.

The strength criterion for the interface $A'\,A$, $y = 0$, $|x| < B/2$ is written as:

(6) $$f\,(\sigma_{xy},\ \sigma_{yy}) = \operatorname{Sup}\,(|\sigma_{xy}| - C,\ \sigma_{yy}) \le 0$$

The halfspace is subjected to the following boundary conditions:

- zero displacement at infinity

(7) $$y \le 0, \qquad |x| \to \infty : \underline{U} = 0$$

- stress free boundary surface outside the foundation $A'\,A$

(8) $$y \le 0, \qquad |x| > B/2 : \sigma_{xx} = \sigma_{xy} = 0.$$

The external load, in addition to the gravity field, is applied on the upper face of the interface $A'\,A$ by the rigid foundation which enforces the following boundary condition for $y = 0^+$, $|x| \le B/2$: the velocity field $\underline{U}$ must be a rigid body motion in the plane Oxy defined by its components in 0: $\underline{U}_0$ and $\underline{\omega} = -\omega\, \underline{e}_z$.

The loading parameters of the problem namely $N\,(\underline{\underline{\sigma}})$, $T\,(\underline{\underline{\sigma}})$, $M\,(\underline{\underline{\sigma}})$ and γ for any statically admissible stress field, together with the associated kinematic ones for any kinematically admissible velocity field have been detailed in the companion paper.

The foundation bearing capacity is given by the boundary of the surface which, in the space (N, T, M), given γ and C, delineates the set of all values for the parameters N, T, M, for which the equilibrium of the foundation is ensured without violating the strength criteria (5) and (6).

In this paper, both the static approach from "inside" and the kinematic approach are used to determine bounds for the foundation bearing capacity. The use of the static approach has been described in detail in the companion paper together with that of the kinematic approach, which requires the introduction of the concept of maximum resisting work.

It is recalled that the former approach yields lower bound estimates for the bearing capacity, the latter, upper bound estimates.

The expression for the maximum resisting work in the soil medium presented in the companion paper must be changed. Referring to [Salençon, 1983, 1993], the expressions for the corresponding densities of maximum resisting work, $\pi\,(\underline{\underline{d}})$ and $\pi\,(\underline{n},\ [\![\underline{U}]\!])$, for any plane strain velocity field, in the case of a purely cohesive soil without tensile strength, are:

$$(9)\qquad \begin{cases} \pi(\underline{\underline{d}}) = +\infty & \text{if } \operatorname{tr}\underline{\underline{d}} < 0 \\ \pi(\underline{\underline{d}}) = C\,|d_1| + |d_2| - \operatorname{tr}\underline{\underline{d}}) & \text{if } \operatorname{tr}\underline{\underline{d}} \geq 0 \end{cases}$$

$$(10)\qquad \begin{cases} \pi(\underline{n},\, [\![\underline{U}]\!]) = +\infty & \text{if } [\![\underline{U}]\!]\cdot\underline{n} < 0 \\ \pi(\underline{n},\, [\![\underline{U}]\!]) = C\,(|[\![\underline{U}]\!]| - [\![\underline{U}]\!]\cdot\underline{n}) & \text{if } [\![\underline{U}]\!]\cdot\underline{n} \geq 0 \end{cases}$$

The kinematic approach then states that, if $\underline{U}$ is a kinematically admissible velocity field with the kinematic data $\underline{U}_0$ and ω, the inequality

$$(11)\qquad -N\,(U_0)_y + T\,(U_0)_x + M\,\omega \leq \int_\Omega \pi(\underline{\underline{d}})\,d\Omega + \int_\Sigma \pi(\underline{n},\, [\![\underline{U}]\!])\,d\Sigma + \int_{A'A} \pi([\![\underline{U}]\!])\,dx + \int_\Omega \gamma\,U_y\,d\Omega$$

yields an upper bound for the foundation bearing capacity (N, T, M).

3. Fundamental results

3.1. Stabilizing effect of the gravity forces on the foundation bearing capacity

In the companion paper, it was recalled that the bearing capacity of the considered foundation did not depend on the unit weight of the soil medium. This result refers to a classical proof based upon the fact that Tresca's strength criterion is a function of the stress deviator only. In the present case, where the strength criterion of the soil exhibits a tension cut-off, the same reasoning can be followed, but the conclusion will be different.

Let $\underline{\underline{\sigma}}^0$ be a stress field in equilibrium with zero gravity forces in the soil medium ($\gamma = 0$) and complying with the strength criteria (5) and (6), and consider the stress field $\underline{\underline{\sigma}}^\gamma$ defined by:

$$(12)\qquad \forall y \leq 0,\quad \forall x;\qquad \underline{\underline{\sigma}}^\gamma(x,\, y) = \underline{\underline{\sigma}}^0(x,\, y) + \gamma\, y\, \underline{\underline{1}},\quad \gamma \geq 0.$$

It is clear, from Eq. (12), that $\underline{\underline{\sigma}}^\gamma$ is in equilibrium with the gravity forces $\underline{\gamma} = -\gamma\,\underline{e}_y$. Since $\gamma\, y \leq 0$ and $\underline{\underline{\sigma}}^0$ complies with the strength criterion (5) in the soil medium, so does $\underline{\underline{\sigma}}^\gamma$. Since $\underline{\underline{\sigma}}^\gamma(x,\, 0) = \underline{\underline{\sigma}}^0(x,\, 0)$, $\underline{\underline{\sigma}}^\gamma$ obviously satisfies the strength criterion (6) in the interface; and $\underline{\underline{\sigma}}^\gamma$ and $\underline{\underline{\sigma}}^0$ equilibrate the same values of the strength parameters then:

$$(13)\qquad N(\underline{\underline{\sigma}}^\gamma) = N(\underline{\underline{\sigma}}^0),\qquad T(\underline{\underline{\sigma}}^\gamma) = T(\underline{\underline{\sigma}}^0),\qquad M(\underline{\underline{\sigma}}^\gamma) = M(\underline{\underline{\sigma}}^0)$$

It follows, from the static approach, that the bearing capacity in the case of non zero gravity forces is greater than or at least equal to the bearing capacity for zero gravity forces.

Contrary to the case of the companion paper, Eq. (12) does not provide an exhaustive construction of all the stress fields $\underline{\underline{\sigma}}^\gamma$ which comply with the strength criteria (5) and (6) and are in equilibrium with the gravity forces $\underline{\gamma} = -\gamma\,\underline{e}_y$. It means that the bearing capacity on a purely cohesive soil without tensile strength may, under certain circumstances, depend upon the soil unit weight which acts as a *stabilizing* factor.

Referring to the kinematic approach, it can also be stated that any upper bound estimate for the bearing capacity on a purely cohesive soil still remains valid for a soil with the same shear strength but without tensile strength. It follows that such an upper bound estimate is valid whatever the gravity forces in the soil medium.

The derivation of additional (and better) upper bounds for the bearing capacity on a soil without tensile strength will proceed from the implementation of velocity fields in equality (11) which take advantage of the new expressions (9) and (10) for the maximum resisting work within the soil. Such velocity fields exhibit dilation ($\operatorname{tr}\underline{\underline{d}} > 0$) and/or uplift velocity jumps ($[\![\underline{U}]\!]\cdot\underline{n} > 0$): consequently, the work of the gravity forces within the soil medium is always negative (the result is established using Stokes' formula) which proves that the unit weight of the soil acts as a stabilizing factor for the upper bound estimates (except for the case when the velocity field only exhibits velocity jumps at the soil boundary below the interface $A'A$ and is non dilatant anywhere else).

As a consequence of the preceding considerations, the problem will be studied assuming zero gravity forces, both for the static and the kinematic approaches. Due to the form of the implemented velocity fields, the computed upper bounds will prove to be valid, even without this assumption, while the lower bounds might be conservative.

3.2. Convexity and method of the reduced width foundation

The general results regarding the convexity of the lower bound estimates of the bearing capacity, and the application of the method of the reduced width foundation presented in the companion paper, remain valid without any alteration.

4. Foundation bearing capacity for an inclined eccentric load on a purely cohesive soil without tensile strength

Since the strength domain defined by Eq. (5) for a purely cohesive soil without tensile strength is contained within the strength domain for a classical purely cohesive soil (as defined in the companion paper), the following general statement holds [Salençon, 1983]:

– for a given soil unit weight γ, the foundation bearing capacity (N, T, M) for a purely cohesive soil with tension cut-off is lower than the foundation bearing capacity for a classical purely cohesive soil (the proof is straigthforward and stems from the static approach from inside).

Moreover, as previously stated, for a soil without tensile strength, the bearing capacity may depend upon the soil unit weight acting as a stabilizing factor: for increasing values of γ, the bearing capacity increases but remains bounded by the bearing capacity of the foundation for a classical purely cohesive soil.

The object of the following study is twofold.

The kinematic mechanisms described for the purely cohesive soil are reanalyzed to introduce slight modifications in order to take advantage of expressions (9) and (10) for the maximum resisting work and therefore yield new, better, upper bounds for the bearing capacity. New mechanisms will subsequently be introduced.

Thereafter, the stress fields of the static approach used for the classical purely cohesive soil will be checked with respect to the strength criterion with tension cut-off (5). New stress fields, compatible with the new criterion, are constructed.

4.1. Kinematic approach

As previously stated, the upper bound estimates described in the companion paper which were derived from the kinematic mechanisms and do not depend upon the soil unit weight, remain valid for the soil without tensile strength.

The associated velocity fields do not imply any volume change within the soil medium:

$$(14)\qquad \begin{cases} \operatorname{tr} \underline{\underline{d}} = 0 \\ [\![\underline{U}]\!] \cdot \underline{n} = 0 \end{cases}$$

The interface $A'A$ incorporates, for the mechanisms A and C, an inclined (not tangential) velocity discontinuity $[\![\underline{U}]\!]$ which corresponds to a separation between the two sides, that is, the soil and the foundation: the discontinuity takes place within the interface. The analysis focuses on this specific aspect.

The new mechanisms A_0 and C_0 are defined below, given that:

– the velocity is identical to that of the corresponding mechanisms A or C within the soil medium;

– the velocity discontinuity, equal to the velocity discontinuity in the interface $A'A$ for the A or C mechanisms, takes place *within the soil immediately below the interface.*

Consequently, when the new mechanisms A_0 and C_0, are compared with the original mechanisms A or C, it is found that

– the work of the external loads remains unchanged and independent of the soil unit weight,

– the maximum resisting work is altered using Eq. (10) to reflect the contribution of the velocity discontinuity along IA (or aA). The maximum resisting work is smaller than or equal to the values derived for the purely cohesive soil. The A_0 and C_0 mechanisms consequently yield better upper bounds for the bearing capacity than the A or C mechanisms.

• *Mechanism* A_0

Referring to Figure 2, the velocity discontinuity $[\![\underline{U}]\!]$ develops along $A'I$ in the soil immediately below the interface. Therefore, the maximum resisting work in that mechanism includes a contribution along a discontinuity line within the soil (whose normal is $\underline{e}_y$) instead of a contribution within the interface.

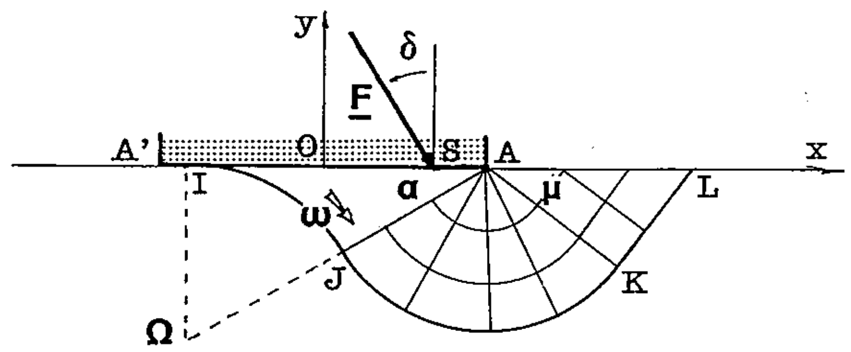

Fig. 2. – Mechanism A_0.

Using the notation presented in the companion paper, the corresponding upper bound estimate for the bearing capacity is given by:

$$(15)\qquad \frac{N}{CB} \leq \lambda^2 \frac{\left\{ \begin{array}{l} \frac{\pi}{4} + \frac{1}{2} + \left(\frac{\pi}{2} - \alpha\right) / \cos^2 \alpha \\ \quad + \tan^2 \alpha \left[1 - \tan^2\left(\frac{\pi}{4} - \frac{\varepsilon}{2}\right) - \operatorname{Ln}\left(\tan^2\left(\frac{\pi}{4} - \frac{\varepsilon}{2}\right)\right)\right]/4 \end{array} \right\}}{\lambda\,(1 + \tan\delta \tan\alpha) + e/B - 1/2}$$

in which the angle ε is defined by:

$$(16)\qquad \tan\varepsilon = (A'I/\Omega I) = (1 - \lambda)/\lambda \tan\alpha, \qquad 0 < \varepsilon < \frac{\pi}{2}$$

The right hand side of inequality (15) is minimized, for a fixed δ, with respect to the parameters λ and α under the same constraints that were applied to mechanisms A:

$$(17)\qquad (1/2 - e/B)/(1 + \tan\alpha \tan\delta) < \lambda \leq 1$$

• *Mechanism* C_0

Referring to Figure 3, the velocity discontinuity $[\![\underline{U}]\!]$ for the C_0 mechanism, develops along $A'a$ *within the soil* immediately below the interface. The application of the same modifications used in mechanisms A leads to the inequality

$$(18)\qquad \frac{N}{CB} \leq \lambda^2 \frac{\left\{ \begin{array}{l} \frac{2\alpha}{\sin^2\alpha} + \left[\operatorname{Ln} \left| \tan\left(\frac{\pi}{4} - \frac{\varepsilon}{2}\right)/\tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) \right| \right. \\ \qquad \left. + \operatorname{Arctan}(\sin\varepsilon) - \operatorname{Arctan}(\sin\alpha)\right] \frac{1}{2\tan^2\alpha} \end{array} \right\}}{\lambda\,(1 + \tan\delta/\tan\alpha) + e/B - 1/2}$$

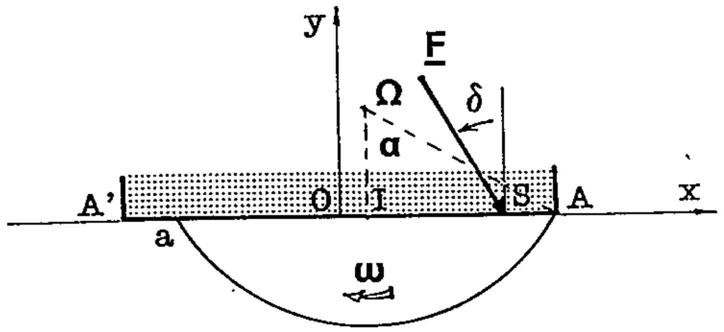

Fig. 3. – Mechanism C_0.

where the angle ε is defined, using the parameters λ and α, by:

$$\tan\varepsilon = (A'I/\Omega I) = (1-\lambda)/\lambda \tan\alpha, \qquad 0<\varepsilon<\pi/2 \tag{19}$$

The right hand side is minimized, for a fixed δ, with respect to λ and α under the same constraints which were applied to mechanisms C

$$\begin{cases} 0<\alpha<\pi/2 \\ 0<\lambda<1/2, \qquad (1/2-e/B)/(1-\tan\delta/\tan\alpha)<\lambda \end{cases} \tag{20}$$

• *Mechanism* F_0

With one exception, the other mechanisms implemented in the companion paper, namely the B and D mechanisms, do not involve a velocity discontinuity along the interface $A'A$. Therefore, they are not affected by the possibility of locating this discontinuity within the soil medium and consequently of improving the corresponding upper bounds. The exception concerns the limiting case of the "unilateral mechanism" examined in the companion paper, which reduced to a slip in the interface $A'A$ when the velocity $\underline{V}$ was tangential to the surface ($\chi = \pi/2$) and to an uplift velocity jump in the interface when $\pi/2 < \chi < \pi$.

In the present case, the homologous mechanism can be developed, assuming that the velocity jump takes place within the soil medium immediately below the interface $A'A$ (*Fig.* 4):

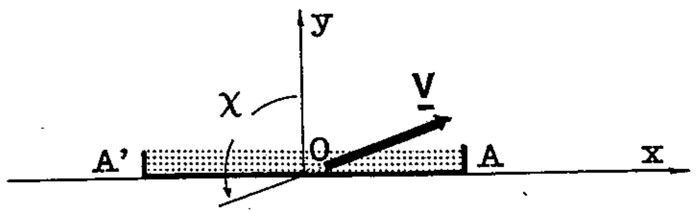

Fig. 4. – Mechanism F_0.

– the foundation follows a translation rigid body motion with the velocity $\underline{V}$ defined by its inclination χ;

– the velocity is continuous across the interface $A'A$;

– a constant velocity discontinuity $[\![\underline{U}]\!] = \underline{V}$ develops within the soil immediately below the interface $A'A$;

– the soil medium ($y<0$) is motionless.

The maximum resisting work in the mechanism is obtained through Eq. (10). For any prescribed value of χ, the corresponding upper bound for the bearing capacity is given by:

$$\begin{cases} \pi/2<\chi<\pi \\ N\cos\chi + T\sin\chi \le CB(1+\cos\chi) \end{cases} \tag{21}$$

The right hand side of inequality (21) is minimized for a fixed δ ($0\le\delta\le\pi/2$) with respect to the parameter χ

$$N/CB \le (1+\cos\chi)\cos\delta/\cos(\delta-\chi) \tag{22}$$

under the constraints:

$$\pi/2 \le \chi \le \delta + \pi/2 \tag{23}$$

The following results are obtained:

– For $\pi/4 \le \delta < \pi/2$, the best mechanism is obtained for $\chi = 2\delta$ and the upper bound for the bearing capacity is:

$$\begin{cases} N/CB = (1+\cos 2\delta) \\ T/CB = \tan\delta\,(1+\cos 2\delta) = \sin 2\delta \end{cases} \tag{24}$$

– For $0\le\delta<\pi/4$, the best mechanism does not depend on δ and is obtained for $\chi=\pi/2$; it yields the upper bound:

$$\begin{cases} N/CB = 1/\tan\delta \\ T/CB = 1 \end{cases} \tag{25}$$

In Figure 5, Eqs. (24) represent a quarter of a circle with a unit radius, centered at ($N/CB = 1$, $T/CB = 0$) and Eqs. (25) stand for the vertical tangent at the abscissa $T/CB = 1$. These upper bounds for the bearing capacity are valid regardless of the load eccentricity e.

• *Summary of the results*

A summary of the results obtained using the kinematic approach with the mechanisms A_0, B, C_0, D and F_0, and their symmetrics B', F'_0, is presented in Figure 5 for load eccentricities $e/B = 0$, 0.1, 0.2, 0.3 and 0.4. The curves delimiting the best upper bounds for the bearing capacity are derived from different mechanisms and the following

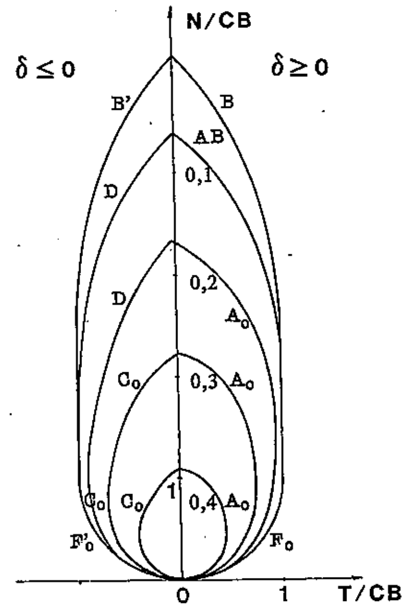

Fig. 5. – Inclined, eccentric load: kinematic approaches from mechanisms A_0, B, C_0, D, F_0 and symmetric ones for various e/B values.

comments, suggested by a comparison with the results in Figure 16 of the companion paper, apply:

– $\mathbf{e/B = 0}$: for $0 \leq |\delta| \leq \pi/4$, the best results are obtained from the unilateral mechanisms of the companion paper (*i.e.* a degenerated mechanism B with $\alpha = \beta$); for $\pi/4 \leq |\delta| \leq \pi/2$, the mechanisms F_0 and F_0' prevail.

– $\mathbf{e/B = 0.1}$: for $\delta \geq 0$, increasing from the vertical loading ($\delta = 0$), the same estimate as for the classical purely cohesive soil is achieved through the mechanisms B (with $\beta = \pi/2$); the associated curve is then extended by an arc obtained from the mechanisms A_0, which is closely coincident with the upper bound obtained by the mechanisms F_0; for $\delta \leq 0$, decreasing from the vertical loading, the same evaluation as for the purely cohesive soil is obtained through the mechanisms D, extended to $\delta = -\pi/2$ by the upper bounds from the mechanisms F_0'.

– $\mathbf{e/B = 0.2}$: for $\delta \geq 0$, the entire curve comes from the mechanisms A_0; for $\delta \leq 0$, the curve is first derived from the mechanisms D then extended by an arc following from the mechanisms C_0, which, for δ values close to $\pi/2$, almost coincides with the upper bound yielded by the mechanisms F_0'.

– $\mathbf{e/B = 0.3}$ **and** $\mathbf{0.4}$: for $\delta \geq 0$, the curves are generated from the mechanisms A_0; for $\delta \leq 0$, from the mechanisms C_0.

Comparing these results with the diagrams of Figure 16 in the companion paper clearly shows a significant decrease in the bearing capacity for the soil without tensile strength. This decrease is more pronounced when the load inclination and/or the load eccentricity increases. The curves will then follow from the mechanisms A_0, C_0 and F_0

which take advantage of the decrease in the maximum resisting work for the soil without tensile strength. From this point of view, the evolution of the leading mechanisms with increasing load eccentricity and load inclination is in good agreement with what could be intuitively expected.

4.2. Static approach from inside for a centered load

It is worth comparing the upper bounds given in Figure 5 for the bearing capacity of the foundation on a cohesive soil with a tension cut-off, with the lower bounds presented in Figure 17 of the companion paper corresponding to a classical purely cohesive soil. It appears that, for high values of load eccentricity or of load inclination, the bearing capacity of the soil with a tension cut-off is (significantly) smaller than the bearing capacity of the classical purely cohesive soil. This proves that, in those cases, the stress fields constructed for the classical purely cohesive soil are definitely not compatible with the strength criterion (5), due to the presence of "unconfined" zones where at least one of the principal stresses is (positive) tensile.

Referring to the comments regarding the influence of gravity forces, it must be recalled that those stress fields are in equilibrium with zero gravity forces within the soil medium and may be substituted for $\underline{\underline{\sigma}}^0$ in Eq. (12). Assuming that such a stress field, defined for $y \leq 0$, incorporates unconfined zones, Eq. (12) shows that, provided that the minimum depth of those zones is non zero and the tension remains finite, the incorporation of the soil unit weight $\gamma > 0$ in γy in the expression for $\underline{\underline{\sigma}}^\gamma$ may balance the tractions in the field $\underline{\underline{\sigma}}^0$: the stress field $\underline{\underline{\sigma}}^\gamma$ will then become compatible with the strength criterion (5) for the soil with a tensile strength cut-off.

This is the stabilizing effect of the gravity forces stated in Paragraph 3.1, the application of which requires that the stress fields for the classical purely cohesive soil be reexamined in order to detect the presence of unconfined zones. The stress fields in equilibrium with centered loads will be checked first, before the application of the method of the reduced width foundation and the use of convexity properties.

• *Stress field in equilibrium with an axial load*

The stress field $\underline{\underline{\sigma}}_s\,(\pi/2)$ in Figure 6 is composed of the Prandtl stress field in the area located above the line $D'C'BCD$ and of the Shield extension below.

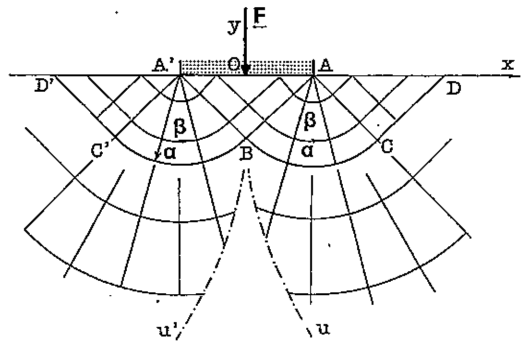

Fig. 6. – Prandtl's stress field and Shield's extension.

The formulation of the first stress field is straightforward and shows that the principal stresses are compressive everywhere.

The Shield stress field is detailed in the original publication [Shield, 1954] and in [Philips, 1956] (*see* also [Salençon, 1969, 1973]). It is composed, in the areas located left of Bu' and right of Bu, of the same stress fields as in areas $A'BC'$ and $A'C'D'$, and ABC and ACD respectively: the principal stresses are again either positive or nil everywhere. In the area located in between the lines of discontinuity Bu' and Bu, the principal directions are Ox and Oy; σ_{xx} (resp. σ_{yy}) does not depend upon x (resp. y); the lines of discontinuity Bu' and Bu and the stresses σ_{xx} and σ_{yy} are determined from the continuity of the stress vector and from the additional condition $\sigma_{yy} - \sigma_{xx} = 2C$ on Bu' and Bu.

Eqs. (26), with $\alpha = \pi/2$, are derived and indicate that the stresses σ_{xx} and σ_{yy} are either compressive or zero everywhere. Consequently, the stress field in Figure 6 which is in equilibrium with the axial load

$$N/CB = \pi + 2, \qquad T/CB = 0, \qquad M = 0$$

and complies with the tension cut-off strength criterion (5).

It follows that this load is also the exact value for the bearing capacity of an axially loaded soil with a tensile strength cut-off.

• *Stress field in equilibrium with an inclined centered load*

The derivation of the stress field in Figure 7 is based on the solution for the bearing

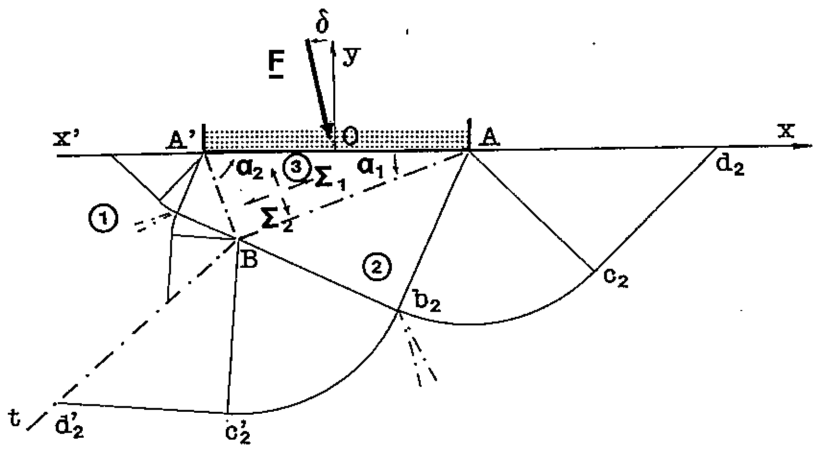

Fig. 7. – Bearing capacity under inclined load: static approach.

capacity on an infinite trapezoidal wedge whose stress field $\underline{\underline{\sigma}}_s(\alpha)$ is depicted in Figure 8. It is composed of the Prandtl stress field whose principal stresses are always either compressive or nil and of its extension by Shield's method, whose description has been given above. With the notations of Figure 8, the following expressions are derived on

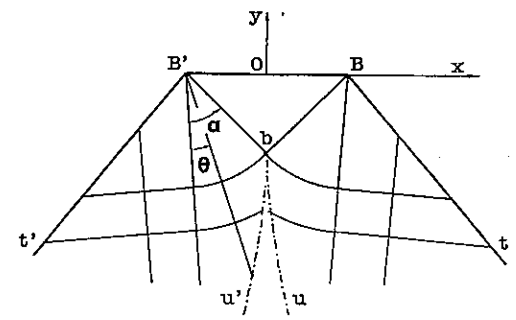

Fig. 8. – Infinite trapezoidal wedge with apex angle 2α: Prandtl's stress field and Shield's extension.

bu' for σ_{xx} and σ_{yy} (Eq. 26) which determine the stress field $\underline{\underline{\sigma}}_s(\alpha)$ in the area in between bu and bu'.

$$(26) \qquad \begin{cases} \sigma_{xx} = 2C\,[\cos(\alpha - \theta) - \theta - 1] \\ \sigma_{yy} = 2C\,[\cos(\alpha - \theta) - \theta] \end{cases}$$

In the extended area, the principal stresses above the lines of discontinuities bu and bu' are either compressive or nil everywhere. In between these lines, Eqs. (26) show that σ_{xx} is always compressive and that:

– if $0 < \alpha < 1$, σ_{yy} is everywhere bounded and tensile ($< 2C$),

– if $1 \leq \alpha < \pi/2$, σ_{yy} is compressive within a "column" centered on the Oy axis which spreads out when α increases until it occupies the entire area in between bu and bu' when $\alpha = \pi/2$. Outside this column, σ_{yy} is bounded and tensile ($< 2C$).

It follows that the stress field $\underline{\underline{\sigma}}_s(\alpha)$, for $0 < \alpha < \pi/2$, is not a valid solution for a soil with a tension cut-off.

Referring now to the stress fields of Figure 7, three areas must be checked. It turns out that these stress fields never comply with the strength criterion (5):

a) for those associated with the arcs $0 \leq N/CB < 1$, $|T|/CB = 1$ of the lower bound estimate in the companion paper, the area 3 is on the whole unconfined, and so are some subareas of areas 1 and 2;

b) for those associated with the remaining parts of arc A, with arc B and arc C, areas 1 and 2 are, in part, unconfined.

Consequently, the stress fields, of the $\underline{\underline{\sigma}}^0$ type, introduced in Paragraph 3.1 of the companion paper, can no longer be used in a static approach. In the first case, the result holds for any $\underline{\underline{\sigma}}^\gamma$ field, as derived through Eq. (12). In the second one, the stabilizing effect of the gravity forces can be effective: the stress field $\underline{\underline{\sigma}}^\gamma$, derived from a stress field $\underline{\underline{\sigma}}^0$, can be used if $\gamma B/C$ is sufficiently large.

• *Consequences for the bearing capacity*

The following conclusions can be drawn from these results:

– the bearing capacity is exactly determined in the case of an axial load ($\delta = 0$), since both the kinematic and the static approaches yield the values $N/CB = (\pi + 2)$, whatever the tension strength of the cohesive soil;

– for $0 < |\delta| \leq \pi/4$, although the stress fields of Figure 7 are no longer compatible with the strength criterion, it is not possible to make a conclusion regarding the instability of the corresponding loads for they might be balanced by other stress fields which comply with the criterion. Moreover, it has been proven that these stress fields are stabilized by the soil unit weight. In the following, attempts are made to construct new stress fields making the application of the lower bound approach possible with the strength criterion (5).

For $|\delta| > \pi/4$, the kinematic approach of Paragraph 4.1 proves that the diagram is no longer valid.

• *New stress field in equilibrium with an inclined centered load*

Khosravi and Salençon [Khosravi, 1983] considered the discontinuous stress field with three zones, presented in Figure 9.

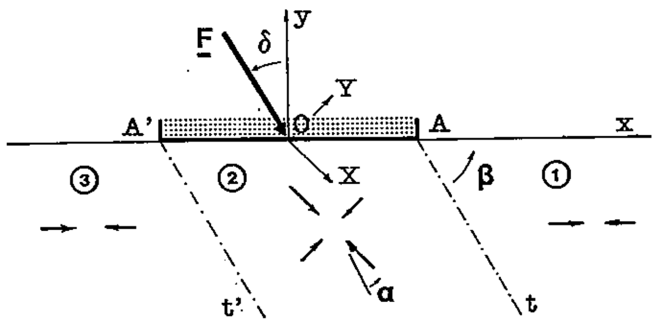

Fig. 9. – Stress field in equilibrium with a centered, inclined load.

The angle β represents the inclination of the lines of discontinuity $A't'$ and At: $0 \leq \beta \leq \pi/2$.

In the areas 1 and 3, the principal stresses are:

(27) $$\sigma_{xx} = -2\,C_1, \qquad \sigma_{yy} = 0, \qquad 0 \leq C_1 \leq C$$

In the area 2, the principal directions are OX and OY defined by the inclination α of OX on At; the principal stresses are:

(28) $$\begin{cases} \sigma_{XX} = -p - C_2, \qquad 0 \leq C_2 \leq C \\ \sigma_{YY} = -p + C_2, \end{cases}$$

The continuity of the stress vector determines α and p as functions of the three independent parameters β, C_1 and C_2:

(29) $$\begin{cases} \sin 2\,\alpha = (C_1/C_2)\ \sin 2\,\beta \\ p = C_1\,(1 - \cos 2\,\beta) + C_2 \cos 2\,\alpha \end{cases}$$

This stress field is in equilibrium, for the weightless soil, with the centered load:

(30) $$\begin{cases} N/CB = (1 - \cos 2\,\beta)\,(C_2 \cos 2\,\alpha - C_1 \cos 2\,\beta)/C \\ T/CB = \sin 2\,\beta\,(C_2 \cos 2\,\alpha - C_1 \cos 2\,\beta)/C \end{cases}$$

whose inclination is $\delta = (\pi/2 - \beta)$, *i.e.* the force $\underline{F}$ is parallel to the lines of discontinuity $A't'$ and At.

In view of the constraints set on C_1 and C_2, these fields obviously comply with the strength criterion for the purely cohesive soil.

Using the static approach with these stress fields, which depend upon three parameters, leads to the determination of the convex envelope of the loads (30) while the parameters β, C_1/C, C_2/C vary under the constraints already listed:

$$0 \leq \beta \leq \pi/2, \qquad 0 \leq C_1/C \leq 1, \qquad 0 \leq C_2/C \leq 1$$

This envelope is defined by the equations for the arcs A, B and C (see *Fig.* 10):

Arc A

(31) $$\begin{cases} 0 \leq \beta \leq \pi/4 \quad \text{hence } \pi/2 \geq \delta \geq \pi/4 \\ C_1/C = 0 \quad (\alpha = 0), \quad C_2/C = 1 \\ N/CB = (1 - \cos 2\,\beta) = 1 + \cos 2\,\delta \\ T/CB = \sin 2\,\beta = \sin 2\,\delta \end{cases}$$

Arc B

(32) $$\begin{cases} \pi/4 \leq \beta \leq 3\,\pi/8 \quad \text{hence } \pi/4 \geq \delta \geq \pi/8 \\ C_1/C = -1/\tan 2\,\beta \quad (\alpha = \beta = -\pi/4), \quad C_2/C = 1 \\ N/CB = \tan \beta \\ T/CB = 1 \end{cases}$$

Arc C

(33) $$\begin{cases} 3\,\pi/8 \leq \beta \leq \pi/2 \quad \text{hence } \pi/8 \geq \delta \geq 0 \\ C_1/C = 1 \quad (\alpha = \pi/2 - \beta), \quad C_2/C = 1 \\ N/CB = -2 \cos 2\,\beta\,(1 - \cos 2\,\beta) = 2 \cos 2\,\delta\,(1 + \cos 2\,\delta) \\ T/CB = -\sin 4\,\beta = \sin 4\,\delta \end{cases}$$

When these stress fields are analysed with regards to the strength criterion (5) for a soil with a tension cut-off, it is found that the most sensitive stress, in area 2, is σ_{YY} which vanishes on the arc A and is compressive on the arcs B and C. In contrast, σ_{yy} is always nil in areas 1 and 3. It follows that the arcs A, B and C in Figure 10 effectively yield a lower bound for the bearing capacity of an inclined centric load on a soil with a tension cut-off.

The accuracy of the lower bound is improved by taking advantage of convexity properties using the exact value of the bearing capacity for an axial load, yielding the arc D in Figure 10.

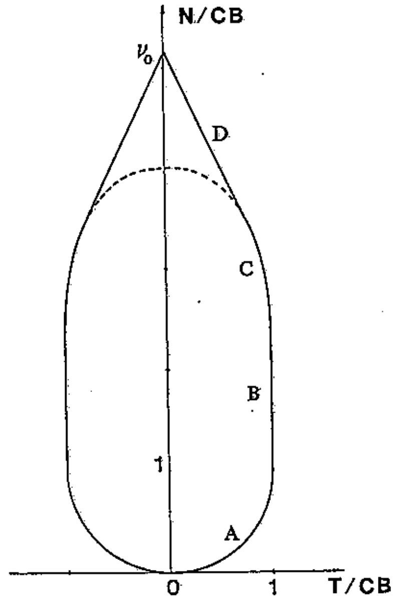

Fig. 10. – Centered, inclined load: convex envelope of the lower bound estimates.

• *Synthesis of the results for a centered load*

Comparing these results with those using the kinematic approaches presented in Figure 5, it appears that for $e/B = 0$, with the exception of the case of the axial load ($\delta = 0$) which has already been mentioned, both approaches give identical results for $\pi/8 \leq |\delta| \leq \pi/4$ in the form of the vertical segments:

$$\pi/4 \leq N/CB \leq 1 + \sqrt{2}, \qquad |T|/CB = 1$$

and for $\pi/4 \leq |\delta| \leq \pi/2$ where they both yield the same quarter of a circle whose equation is given by (24) or (31) and its symmetric. The bearing capacity is computed exactly.

The identity of the two results reveals that the mechanisms F_0 considered in the kinematic approach and the stress fields which define the arcs A and B in Figure 10, are associated with each other as shown in Figure 11. For $\pi/4 \leq |\delta| \leq \pi/2$, the foundation rests on the "soil column" $t'\,A'\,A\,t$ (area 2) which is in unconfined compression parallel to the force $\underline{F}$. The tension cut-off in the strength criterion makes a velocity discontinuity

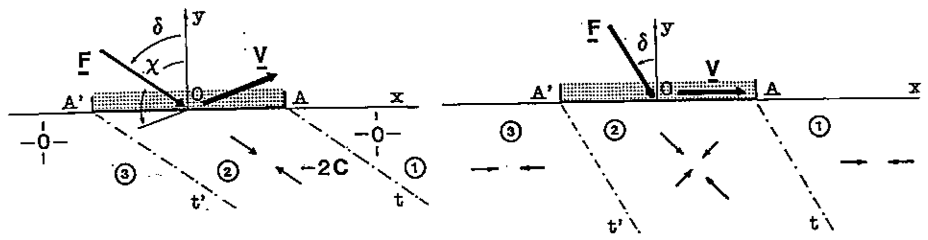

Fig. 11. – Velocity field F_0 and associated stress field.

$[\![U]\!] = \underline{V}$ possible in the soil medium immediately below the interface $A'\,A$, acting at an inclination $\chi = 2\,\delta$.

For $\pi/8 \leq |\delta| \leq \pi/4$, the soil column parallel to the force $\underline{F}$ in area 2 is laterally confined by areas 1 and 3; the principal stress directions are inclined at $\mp\pi/4$ on Ox, and the velocity field is a slipping mechanism within the soil below the interface $A'\,A$, and parallel to $A'\,A$.

Referring to Figure 10 and recalling the stabilizing effect of the gravity forces within the soil medium on the stress fields of Figure 7, it also follows that, when $\gamma\,B/C$ is large enough, the lower bound estimate of the bearing capacity for $0 < \delta < \pi/8$ is improved: the lower bound obtained for the classical purely cohesive soil can then be retained for the cohesive soil with a tension cut-off.

4.3. Static approaches for an eccentric load

The lower bounds for eccentric loads are obtained using the reduced width foundation method and taking advantage of the convexity properties in the (N, T, M) space stated in Paragraph 3.2.

The results are presented in Figure 12 where it appears that the convexity property, which has been numerically implemented, does improve the lower bound estimate: this improvement is particularly noticeable for the maximum value of T/CB, for a fixed e/B.

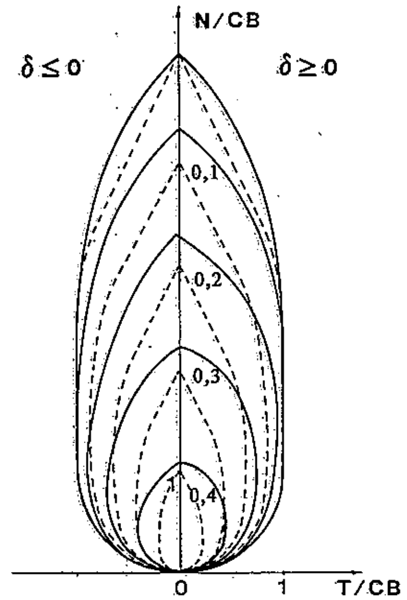

Fig. 12. – Bearing capacity under inclined eccentric load.
Solid lines: upper bound estimates. Dotted lines: lower bound estimates.

4.4. Bearing capacity for an eccentric inclined load on a purely cohesive soil with a tensile strength cut-off

Figure 12 compares the results of both approaches showing the upper and lower bound estimates for the bearing capacity for each indicated value of e/B. This diagram may be compared with the corresponding one for the classical purely cohesive soil.

As stated previously, for a centered load ($e/B = 0$), the foundation bearing capacity on a soil with a tension cut-off is determined exactly for $\delta = 0$ and for $\pi/8 \leq \delta \leq \pi/2$. Between 0 and $\pi/8$, the assessment of the bearing capacity is obtained with less than a 4% error for $\gamma B/C = 0$, an accuracy which improves as $\gamma B/C$ increases.

The comparison with the results obtained for the classical purely cohesive soil for the centered load ($e/B = 0$) shows that, for $\delta = 0$ and $\pi/8 < \delta < \pi/4$, the bearing capacities are equal to each other, regardless of the strength criterion, they do not depend upon the soil unit weight; for $0 < \delta < \pi/8$, the difference, if any, between both values, does not exceed 7 to 8% for $\gamma B/C = 0$. The most significant difference appears for $\pi/4 < |\delta| < \pi/2$. As a matter of fact, within this range of δ, the bearing capacity diagram is nothing but the representation, using axes (N/CB, T/CB), of the *comprehensive criterion* for the interface $A'A$ [Salençon, 1983], *i.e.* of the criterion expressed in terms of σ_{yy} and σ_{xy} which simultaneously takes into account the limitations imposed by the interface itself and by the constituent materials on either side of the interface (in the present case, only the soil is considered, since the foundation is assumed to be rigid). This is not merely a coincidence since the mechanisms corresponding to the arcs on the bearing capacity diagram (unilateral mechanism with $\chi = \pi/2$, and mechanism F_0) imply a velocity discontinuity across the interface.

It must be noted that the most important parameter is the tension cut-off in the soil, since in both cases, the interface does not present any resistance in tension.

Referring to Figure 9 in the companion paper, where the diagram obtained from Meyerhof [Meyerhof, 1963], "parabolic" formula is drawn, it appears that for $\pi/3 < |\delta| < \pi/2$, the values obtained using this formula slightly overestimate the bearing capacity.

As the load eccentricity increases, the difference between both bounds increases significantly, despite the slight improvement achieved by utilizing the convexity properties in the results obtained from the reduced width foundation method. Comparing these results with the results for the purely cohesive soil, it appears that:

– the bearing capacities are considerably lower for large load inclinations,

– for $\pi/12 \leq |\delta| \leq \pi/6$, the lower bounds computed for the same e/B values are comparable.

Furthermore, the symmetry of the lower bound diagrams for $e/B \neq 0$ is a result of their construction methods; on the other hand, it may be observed that the upper bound diagrams appear to be more symmetrical in Figure 12 than they did for the classical purely cohesive soil; however, a definite conclusion on that point would be risky.

Finally, it will again be recalled, as explained in the companion paper, that in the case of the interface which exhibits a strength criterion which differs from (6), all the diagrams in Figure 12 should be truncated by the corresponding curves which express the interface strength criterion in terms of N/CB and T/CB instead of σ_{yy} and σ_{xy}: e.g. $|T|/CB \leq N \tan \phi_1/CB$ for a Coulomb strengh criterion with a friction angle ϕ_1. Obviously, this is in line with what has just been stated about the comprehensive strength criterion of the interface.

5. Conclusions

Within the same framework of the yield design theory which was adopted for the case of a classical purely cohesive soil in the companion paper [Salençon & Pecker, 1995], the ultimate bearing capacity of a foundation resting on a cohesive soil with no tensile strength has been studied for an inclined eccentric load.

Due to the strength criterion of the soil, the bearing capacity in the general case of inclination and eccentricity can no longer be proven to be independent from the gravity forces, whose effect will always be a stabilizing one, and is governed by the adimensional factor $\gamma B/C$. However, for the case of a centered load with a small inclination, independence has been proven or may be anticipated.

As a general comment, the effect of the tension cut-off condition can be perceived within the soil medium and in the interface. Therefore, new stress fields were constructed in order to comply with the additional tension cut-off condition. New velocity fields were then implemented which could take advantage of the expression of the maximum resisting work associated with the comprehensive strength criterion of the interface.

The foundation bearing capacity has been computed, exactly or, at least, with a very high accuracy, for a centered load. For an eccentric load the bearing capacity has been bracketed between lower and upper bounds, as a function of the load eccentricity and load inclination. From a practical standpoint, in view of the safety factors used in the design of foundations under vertical, centered, loads, the implications of the inclinations and eccentricities are usually small.

The results presented herein have been used for the seismic analyses of foundations and are reported in [Pecker & Salençon, 1991].

Since both cases studied in this paper and in the companion one, represent extreme conditions for the tensile resistance for the purely cohesive soil, it can be stated that the results bracket the variations of the bearing capacity as a function of the soil tensile strength.

Acknowledgments

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