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Limit state design in geotechnical engineering

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A theoretical approach to the Ultimate Limit State Design

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SYNOPSIS
The present contribution addresses the issue of Ultimate Limit State Design for which a rigorous mechanical formulation based on the yield design theory is proposed. It is shown in particular how the kinematic approach of this theory provides a clear statement of the problem, thus avoiding some shortcomings appearing in classical methods. The performance of this approach which can be applied to any kind of geotechnical structure, is illustrated on the case of soil reinforcement.

1. INTRODUCTION

Expressed in general terms, the principle of the Ultimate Limit State Design may be described as follows (Krebs Ovesen, 1989). "Ultimate limit states involve safety : loss of static equilibrium or rupture of a critical section of the structures (...). According to the principles of Limit States Design, the design criterion is simply to design for equilibrium in the design limit state of failure. The design criterion could be expressed in the following way

$$S_d \leq R_d \quad (1)$$

S_d is the design load effect, (...), R_d is the design resistance effect".

As it appears from the above statement, the concept of U.L.S.D. first requires a clear distinction to be made between the loads applied to the structure and the strengths of its constituent elements.

The design values of the loads (or actions), which are given quantities, are obtained by assigning to each of their characteristic values a multiplier, called partial safety factor. This factor is greater than unity when the effect of the corresponding load is unfavourable to the equilibrium of the structure, smaller than unity in the opposite case.

On the other hand, the design values of the parameters which characterize the resistance of the structure are derived by dividing each of them by an appropriate partial safety factor which is always larger than unity.

The practical implementation of the concept of U.L.S.D implies to quantify the respective "effects" of the design loads and resistances, in order to make it possible to compare them through an inequality such as (1). The yield design theory will be shown to be an (if not the) adequate mechanical framework for performing such a quantification in a rational way.

2. FUNDAMENTALS OF THE YIELD DESIGN THEORY (Salençon, 1983,1990)

The yield design reasoning proceeds from the simple idea that the equilibrium of a structure under its applied loads must be compatible with the strength of its constituents.

Within the usual formalism of classical three dimensional continuum mechanics, a structure is considered with a volume Ω, subjected to several loading parameters $\underline{Q}=(Q_i)$, $i=1,\ldots,n$. The strength capacities of its constituent materials are defined at any point $\underline{x}$ of Ω by the set of allowable stress tensors $\underline{\underline{\sigma}}(\underline{x})$ which may be represented by a convex domain $G(\underline{x})$ in the space of stresses.

A loading $\underline{Q}$ will be termed "safe", if it can be equilibrated by a stress field $\underline{\underline{\sigma}}$ in Ω, which complies with the strength criterion everywhere :

$$Q \in K \Leftrightarrow \begin{cases} \exists \underline{\underline{\sigma}} \text{ in equilibrium with } \underline{Q} \\ \underline{\underline{\sigma}}(\underline{x}) \in G(\underline{x}), \ \forall \underline{x} \in \Omega \end{cases} \quad (2)$$

The domain K defined through Eq. (2) is called the *stability domain* of the structure in the space of loadings. It is *convex* as a consequence of the convexity of the domains $G(\underline{x})$. Its construction may be achieved by exploring all the possible statically admissible stress fields which satisfy the strength requirement. This is the *static approach* of the yield design theory. It leads to *lower-bound estimates* for the *extreme* (or ultimate) loads located on the boundary of K.

The stability domain K may also be determined "from the outside" by means of the *kinematic approach* of the yield design theory which is the dualization of the static approach through the principle of virtual work. According to this approach, any safe load $\underline{Q}$ (i.e. which belongs to K) satisfies the following fundamental inequality :

$$W_e(\underline{Q}, \hat{\underline{U}}) \leq W_{mr}(\hat{\underline{U}}), \ \forall \hat{\underline{U}} \quad (3)$$

In that inequality, the term $W_e(\underline{Q}, \hat{\underline{U}})$ is the work of the load $\underline{Q}$ in any kinematically admissible virtual velocity field $\hat{\underline{U}}$, which writes

$$W_e(\underline{Q}, \hat{\underline{U}}) = Q_j \dot{q}_j(\hat{\underline{U}}) = \underline{Q} \cdot \dot{\underline{q}}(\hat{\underline{U}}) \quad (4)$$

according to the definition of the loading parameters.

The term $W_{mr}(\hat{\underline{U}})$ is called the *maximum resisting work*. Its expression writes :

$$W_{mr}(\hat{\underline{U}}) = \int_\Omega \pi(\hat{\underline{\underline{d}}}) \, d\Omega + \int_\Sigma \pi(\underline{n}, [\hat{\underline{U}}]) \, d\Sigma \quad (5)$$

where $\hat{\underline{\underline{d}}}$ is the strain rate tensor associated with $\hat{\underline{U}}$ ($\hat{d}_{ij} = \frac{1}{2}(\partial \hat{U}_i / \partial x_j + \partial \hat{U}_j / \partial x_i)$), $[\hat{\underline{U}}]$ denotes the jump of $\hat{\underline{U}}$ across a velocity discontinuity surface Σ, following its normal $\underline{n}$ and

$$\pi(\hat{\underline{\underline{d}}}(\underline{x})) = \sup\left\{\underline{\underline{\sigma}} : \hat{\underline{\underline{d}}}(\underline{x}) \,;\, \underline{\underline{\sigma}} \in G(\underline{x})\right\} \quad (6)$$

$$\pi(\underline{n}(\underline{x}), [\hat{\underline{U}}(\underline{x})]) = \sup\left\{\underline{n}(\underline{x}) \cdot \underline{\underline{\sigma}} \cdot [\hat{\underline{U}}(\underline{x})], \underline{\underline{\sigma}} \in G(\underline{x})\right\} \quad (7)$$

where $\underline{\underline{\sigma}} : \hat{\underline{\underline{d}}} = \sigma_{ij} \hat{d}_{ji}$, with the convention of summation over repeated subscripts.

Taking account of (4), Ineq.(3) can be given the geometrical interpretation shown in figure 1. In the space of loads $\underline{Q}$, domain K is included in the half space bounded by the plane of equation $\underline{Q}\cdot\dot{\underline{q}}(\hat{\underline{U}}) = W_{mr}(\hat{\underline{U}})$ and containing the origin.

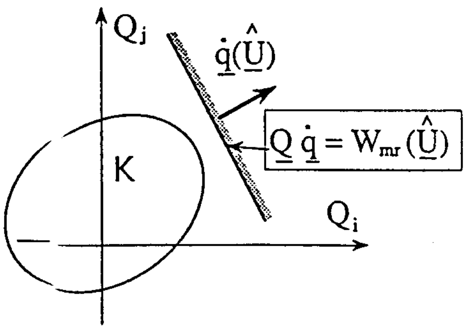


Fig. 1. Domain K of safe loads and its kinematic approach.

3. STATIC FORMULATION OF THE U.L.S.D. (Salençon, 1993)

Each action applied to the structure will correspond to a loading parameter Q_j, and the set of their design values will be denoted by $\underline{Q}(S_d)$. In the same way, the strength properties of the constituent materials of the structure are defined by a set of parameters whose design values will be symbolically denoted by R_d, so that the strength condition at any point $\underline{x}$ of Ω writes :

$$\forall \underline{x} \in \Omega,\quad \underline{\underline{\sigma}}(\underline{x}) \in G(R_d;\underline{x}) \qquad (8)$$

The yield design theory starting from $G(R_d;\underline{x})$ instead of $G(\underline{x})$ makes it possible to define the corresponding stability domain $K(R_d)$ in the loading parameters space. This domain makes it possible to implement the U.L.S.D. concept by considering the different combinations of loads obtained when exploring all possible permutations of the maximum and minimum values of the partial safety factors. As shown schematically in figure 2 in the case of two loading parameters, it amounts to checking that every vertex of a rectangle is included in $K(R_d)$, since the *convexity* of $K(R_d)$ ensures that the rectangle as a whole will belong to $K(R_d)$.

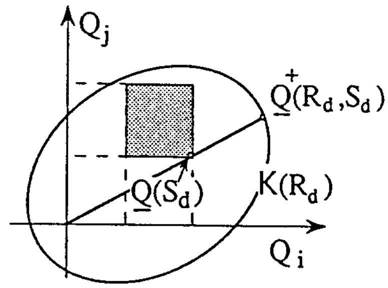


Fig. 2. Static formulation of the U.L.S.D.

Introducing the *factor of confidence* of the structure under its design applied load $Q(S_d)$ defined as :

$$\Gamma(S_d,R_d) = Q^+(R_d,S_d)/Q(S_d) \qquad (9)$$

where $Q^+(R_d,S_d)$ is the extreme load proportional to $Q(S_d)$ (fig. 2), Ineq. (1) may be specified in the form :

$$\Gamma(S_d,R_d) \geq 1 \qquad (10)$$

4. FORMULATION BY MEANS OF THE KINEMATIC APPROACH (Salençon, 1993)

Denoting the design values of the loads and the strength parameters by S_d and R_d respectively , the fundamental inequality (3) can be rewritten in the form :

$$\forall\underline{\hat{U}} \; W_e(S_d,\underline{\hat{U}}) \leq W_{mr}(R_d,\underline{\hat{U}}) \qquad (11)$$

with $W_e(S_d,\underline{\hat{U}})=W_e(Q(S_d),\underline{\hat{U}})$.

As it is clearly apparent from Ineq.(11), the respective "effects" of the design loads and resistances, which up to now referred to a rather vague notion, are now given a precise meaning, namely the work of the design loads on the one hand, the maximum resisting work derived from the design values of the strength parameters on the other hand. Both of them are calculated in the same velocity field $\underline{\hat{U}}$. Ensuring the stability of the structure would require to verify Ineq. (11) for all the kinematically admissible virtual velocity fields. From a practical point of view, the exploration of the velocity fields is usually restricted to a particular class of "failure mechanisms".

In order to account for such a limitation, a coefficient of method Γ_s, greater than unity, may be introduced so that :

$$\Gamma_s W_e(S_d,\underline{\hat{U}}) \leq W_{mr}(R_d,\underline{\hat{U}}) \qquad (12)$$

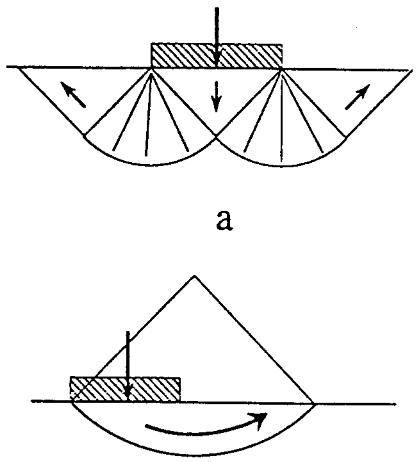
is to be satisfied for all the considered mechanisms.

The example of a classical problem will help to illustrate such an idea. Considering the bearing capacity problem of a strip footing of width B resting on a purely cohesive soil of cohesion C, a comprehensive application of Ineq.(11) to all the kinematically admissible virtual velocity fields leads to the exact value of the bearing capacity:

$$F^+=(\pi+2)BC \qquad (13)$$

obtained for instance with Prandtl's velocity field (figure 3a). A partial exploitation of the same formula making only use of slip circle mechanisms (figure 3b) yields the following upperbound value :

$$F^{+} \leq F^{\star} \simeq 5.53\ BC \qquad (14)$$



a

b

Fig. 3. Bearing capacity of a strip footing. a) Prandtl's mechanism; b) Slip circle mechanism.

Comparing formulae (13) and (14) suggests that a coefficient of method equal to $F^{\star}/F^{+} \simeq 1.08$ or so, could lead to the same level of safety in a "slip circle" analysis as in the complete one.

5. APPLICATION TO THE DESIGN OF REINFORCED SOIL STRUCTURES

Shortcomings of traditional design methods as regards the requirement of U.L.S.D. are particularly apparent when dealing with complex geotechnical structures such as reinforced earthworks. The above described approach of the yield design theory and more specifically the kinematic method, make a rigorous treatment of this kind of problem possible.

The complexity of designing reinforced soil structures is mainly due to the composite nature of the constituent material (soil + inclusions). In stability analyses using the "method of slices" (Clouterre 1991), a certain ambiguity still remains for instance as to whether the forces in the inclusions should be regarded as actions or resistances in a failure design method: it is apparent from Ineq.(1) that choosing one option or the other leads to completely different answers as concerns safety requirements.

Within the framework of the yield design theory, both the soil and the reinforcements are considered as the constituent materials of the structure with the respective strength parameters : cohesion (C) and friction angle (ϕ) for the soil; ultimate tensile load bearing capacity (N_o) for the reinforcements. A more general strength condition could be adopted involving the shear and bending resistance of the reinforcements (de Buhan and Salençon, 1993). The stability of a retaining structure such as that sketched in figure 4 can be investigated for example by exploring a class of velocity fields in which a volume of reinforced soil (OAB) is given a rigid body motion with angular velocity $\hat{\omega}$ about a point Ω.

The work performed by the external loads applied to the structure (weight of the soil, distributed or concentrated forces acting upon the surface, seepage forces,...) in such a failure mechanism writes (de Buhan et al.,1992a)

$$W_e(\underline{Q},\hat{\underline{U}}) = M_e^{\Omega}(Q_1)\hat{\omega} \qquad (15)$$

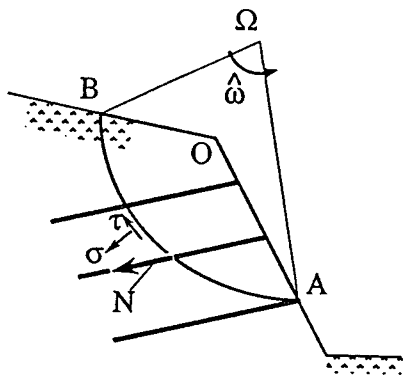


Fig. 4. Stability analysis of a reinforced soil retaining structure.

where $M_e^{\Omega}(Q_1)$ may be called the "driving moment", i.e. the moment about Ω of the given external loads applied to OAB.

The *maximum resisting work* takes the form :

$$W_{mr}(\hat{\underline{U}})=M_{mr}^{\Omega}(C,\tan\phi,N_o,|\hat{\omega}|/\hat{\omega})\hat{\omega} \qquad (16)$$

where M_{mr}^{Ω} represents the *maximum resisting moment* about the same point due to the respective strength capacities of both the soil and the reinforcements defined by the parameters C, tan ϕ and N_o. It is worth noticing that M_{mr}^{Ω} depends on the sign of $\hat{\omega}$ which appears through $|\hat{\omega}|/\hat{\omega}$ and that $W_{mr}(\underline{\hat{U}})$ is always positive.

Consequently, the fundamental inequality (3) reduces to the following necessary stability condition, interpreted in terms of moments :

$$\pm M_e^{\Omega}(Q_i) \leq \pm M_{mr}^{\Omega}(C, \tan\phi, N_o, \pm 1) \quad (17)$$

or introducing the partial safety factors γ_{Qi} attached to the loads Q_i, as well as those relative to the strength parameters $\gamma_C, \gamma_\phi, \gamma_{No}$

$$\pm M_e^{\Omega}(\gamma_{Qi} Q_i) \leq \pm M_{mr}^{\Omega}(C/\gamma_C, \tan\phi/\gamma_\phi, N_o/\gamma_{No}, \pm 1) \quad (18)$$

The more constraining of the two inequalities in Form. (18) writes eventually

$$|M_e^{\Omega}(\gamma_{Qi} Q_i)| \leq \left(|M_e^{\Omega}|/M_e^{\Omega}\right) M_{mr}^{\Omega} (C/\gamma_C, \tan\phi/\gamma_\phi, N_o/\gamma_{No}, |M_e^{\Omega}|/M_e^{\Omega}) \quad (19)$$

From this formula it is easy, given curve AB and point Ω, both signs of M_e^{Ω} being considered successively (if necessary), to identify which loads are favourable (resp. unfavourable) to the equilibrium. For a given sign of M_e^{Ω}, a load Q_i whose contribution decreases $|M_e^{\Omega}|$ is favourable to the equilibrium and will therefore be given a partial safety factor smaller than unity; a load whose contribution increases $|M_e^{\Omega}|$ is unfavourable to the equilibrium and will be attributed a partial safety factor greater than unity. By the way, it turns out that most often, in order for $M_{mr}^{\Omega}(C/\gamma_C, \tan\phi/\gamma_\phi, N_o/\gamma_{No}, |M_e^{\Omega}|/M_e^{\Omega})$ to remain finite, only one possibility is available for the sign of M_e^{Ω} for a given curve AB and a given point Ω.

The most constraining condition is then obtained by searching for the minimum value of the ratio $M_{mr}^{\Omega}/M_e^{\Omega}$ with respect to Ω and AB. This is achieved, first by noting that for Ω being kept fixed, the most critical curve AB is an arc of logspiral of angle ϕ_d ($\tan\phi_d = \tan\phi/\gamma_\phi$) anf focus Ω, then by minimizing the above ratio with respect to a few angular parameters which define the geometry of the arc of logspiral (de Buhan et al., 1992a).

As it has been previously observed, Ineq. (19) provides only a *necessary* stability condition, so that it remains advisable to adopt a coefficient of method Γ_s, greater than unity, thus changing Ineq. (19) into

$$\Gamma_s |M_e^\Omega| \leq \left(|M_e^\Omega|/M_e^\Omega\right) M_{mr}^\Omega \qquad (20)$$

This approach, making use of design values derived by applying partial safety factors, is worth being compared to the classical *global safety factor approach.* It turns out that such a global safety factor F, may be defined as the factor by which the sole strength parameters of the soil C and $\tan\phi$ are to be divided, so that (19) reduces to an equality

$$M_e^\Omega(Q_i) = M_{mr}^\Omega(C/F, \tan\phi/F, N_o) \qquad (21)$$

In other words, it amounts to taking $\gamma_C=\gamma_\phi=F$, all the other partial safety factors, and the coefficient of method as well being set equal to unity.

6. CONCLUSION

Adopting the U.S.L.D. approach in the field of soil mechanics and geotechnical engineering leads to reconsidering traditional methods. This is particularly apparent when complex structures involving soil reinforcement, for instance, are concerned. It has been shown in this paper that the yield design theory stands as a sound mechanical basis for a rational implementation of U.L.S.D.. Its kinematic approach appears as most appropriate and makes it possible to deal with any practical situation without any ambiguity as regards the distinction to be made between resistances and actions (or loads), and the identification of favourable and unfavourable. Moreover, it lends itself to a numerical implementation in the form of efficient computer codes, very easily (de Buhan et al., 1992b).

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