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A Theoretical Approach to Ultimate Limit States Design

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1. The concept

1.1 Governing principles of Ultimate limit states design

Ultimate limit states design (ULSD) is presently tending to serve as a reference for safety analyses in civil and construction engineering, as can be seen, for instance, when looking at the deliberations related to future European codes. As an example, one may quote Krebs Ovesen (1989) and define its principles as follows.

"Ultimate limit states involve safety: loss of static equilibrium or rupture of a critical section of the structures" …

"According to the principles of Limit States Design, the design criterion is simply to design for equilibrium in the design limit state of failure. The design criterion could be expressed in the following way

$$\mathbf{S}_{\mathrm{d}} \le \mathbf{R}_{\mathrm{d}} \tag{1.1}$$

where $\mathbf{S}_{\mathrm{d}}$ is the design load effect,…, $\mathbf{R}_{\mathrm{d}}$ the design resistance effect."

This statement first highlights the fact that the mechanical analysis of the structure under concern must make a clear distinction between the loads applied to the structure and the resistances of its constituent elements. It also calls for the definition of the design values of these quantities. And finally, the concept of an effect must be clarified.

1.2 Loads and their design values

The concept of loads refers to actions – point-, surface- or volume forces, moments – exerted on the structure, that are given and stand as data. Their design values are obtained through the application to each of them of a partial safety factor, a multiplier that may depend on the considered combination of loads, and is

Superior to 1, when the effect of the load is unfavourable to equilibrium

Inferior to 1, when this effect is favourable.

This latter point underscores the necessity, at this stage, of a precise and mechanically rigorous definition of the concept of effect introduced in the preceding Section.

1.3 Resistances and their design values

Resistances are mechanical characteristics of the structure constituent materials. They are derived from identification experiments and stand as limitations on the interior forces at any point of the structure. In the case of interior forces being modelled as one-dimensional, the resistance of a material element is given through scalar values that stand upper and lower bounds for the intensity of the interior effort: e.g., the resistance to tension and compression of a bar only subjected to an axial force. Generally speaking, when interior forces are modelled as multidimensional, the resistance of a material element is expressed through a criterion, which is the case of classical continuous media, or an interaction formula, as in the case of curvilinear media, that depend on one or several resistance parameters characterizing the element. The design values of these parameters, to be introduced in the resistance criterion, are obtained through the application to each of them of a partial safety factor, which is a divisor superior to 1.

1.4 The effects of loads and resistances

1.4.1 The issue

Defining the concept of an "effect on the equilibrium of the system" is an obvious necessity for a practical implementation of the principles set out in Section 1.1. Equation (1.1) is symbolical and must be made explicit and quantified. This implies that the effect of the loads and the effect of the resistances can be measured, expressed and compared through scalar values corresponding to the same physical reality.

1.4.2 Statically determinate structure of pin-connected elements

Let us examine, first, an example that does not involve any difficulty about this issue: a statically determinate pin-hinged structure that is only loaded and supported at its connecting nodes as shown in Figure 1.

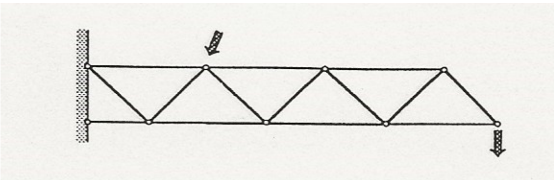

Figure 1. Statically determinate pin-hinged structure

For any given design load combination, denoted by S_{d}, the distribution of interior forces that are in equilibrium with S_{d} in that structure is fully determined: it is a field of tensile or compressive forces in the bars, i.e., a scalar field. It is then consistent to interpret Equation (1.1) as follows:

- In the bars subjected to tension, comparing the tensile force field generated by the design loads S_{d} with the field of design values for the resistances to tension denoted by $(R_{\mathrm{d}}^{\mathrm{T}})$,
- In the bars subjected to compression, comparing the compressive force field generated by the design loads S_{d} with the field of design values for the resistances to compression denoted by $(R_{\mathrm{d}}^{\mathrm{C}})$

In each barre (i), the normal force can be split into its positive and negative parts in the form

$$N_i = N_i^+ - N_i^- \text{ with } N_i^+ \geq 0,\ N_i^- \geq 0,\ N_i^+ N_i^- = 0 \tag{1.2}$$

so that (1.1) can be expressed as

$$\forall i,\ -(R_{\mathrm{d}}^{\mathrm{C}})_i \leq N_i(S_{\mathrm{d}}) \leq (R_{\mathrm{d}}^{\mathrm{T}})_i$$

or, equivalently,

$$\left.\begin{array}{l}\operatorname*{Sup}_{i}\left\{N_i^{+}(S_{\mathrm{d}})-(R_{\mathrm{d}}^{\mathrm{T}})_i\right\}\leq 0\\ \operatorname*{Sup}_{i}\left\{N_i^{-}(S_{\mathrm{d}})-(R_{\mathrm{d}}^{\mathrm{C}})_i\right\}\leq 0\end{array}\right\}. \tag{1.3}$$

1.4.3 Statically determinate structure with multidimensional interior forces

Within the same rationale, we can consider the case of a statically determinate structure where interior forces are modelled as multidimensional, with several scalar fields such as normal force N, bending moment M, etc. The resistance of a constituent element within this framework is defined, at any point of the structure by an interaction formula such as

$$f(R;N,M,...)\leq 0 \tag{1.4}$$

where R denotes the resistance parameters attached to the material element.

Following the same rationale as above, Equation (1.1) amounts to looking for the maximum value of the interaction formula over the structure when the interior forces are generated by the design loads S_{d} with the design values of the resistance parameters R_{d}. As a result, (1.1) will be expressed as

$$\operatorname*{Sup}_{i} f(R_{\mathrm{d}};N(S_{\mathrm{d}}),M(S_{\mathrm{d}}),...)\leq 0 \tag{1.5}$$

which is the counterpart of (1.3).

1.4.4 Statically indeterminate structures and systems

What has been proposed above as a natural answer, obviously relies on the static determinacy of the considered structure. Statically indeterminate structures, and systems modelled within the framework of continuum mechanics, do not allow for this kind of answer, due to the fact that the field of interior forces that are in equilibrium with a given load combination is not uniquely determined.

The governing principles of Ultimate limit states design do not allow for the introduction of any complementary assumption[1], whatsoever, that would make it possible to distinguish a specific field of interior forces among all the fields that are in equilibrium (i.e., statically admissible) with a given combination of loads S_{d}. Let the set generated by these statically admissible fields be denoted by $\mathbb{S}(S_{\mathrm{d}})$, then, a natural interpretation of the governing principles amounts to exploring $\mathbb{S}(S_{\mathrm{d}})$ and checking if, for one field of interior forces in equilibrium with the design load combination, the counterpart of Equation (1.5) is satisfied.

Adopting for simplicity's sake the notations of classical continuum mechanics:

$\underline{\underline{\sigma}}$: field of interior forces,

Ω : volume of the structure under consideration,

$\underline{x}$: current point in Ω,

$f\big(R;\underline{x},\underline{\underline{\sigma}}(\underline{x})\big)\leq 0$: resistance criterion for the material element at point $\underline{x}$,

the explicit formulation of Equation (1.1) will now be written as

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[1] An assumption as to the behaviour of the constituent material, or any data related to all or part of the loading history of the system.

$$\operatorname*{Inf}_{\underline{\underline{\sigma}}\in \mathbf{S}(S_\mathrm{d})} \operatorname*{Sup}_{\underline{x}\in\Omega} f\left(R_\mathrm{d};\underline{x},\underline{\underline{\sigma}}(\underline{x})\right) \leq 0 \tag{1.6}$$

1.5 Comments

It must be observed that Equations (1.3) and (1.6), which are proposed as mathematically precise expressions for a practical implementation of fundamental inequality (1.1), suffer from the drawback that they do not maintain the very structure of (1.1), where two quantities, attached respectively to the design loads and the design resistances are compared to each other. This drawback is all the more apparent when it is found appropriate to modify Equation (1.1) through the introduction of a “method coefficient” Γ_s in the form

$$\Gamma_\mathrm{s}\,\mathbf{S}_\mathrm{d} \leq \mathbf{R}_\mathrm{d} \tag{1.7}$$

In addition, from a practical viewpoint, Equations (1.3) and (1.6) make it difficult to make a distinction between the loads, whose effect is favourable to the equilibrium and those, whose effect is unfavourable. This requires examining all design load combinations, retaining both possibilities for each load and finally retaining the load combination S_d that makes a maximum of the left-hand side term of (1.6).

Besides, one may question about the relevance of a formulation such as (1.6), since it is limited to checking, for any design load combination, whether it is possible to find a statically admissible field that complies with the design resistances.

As a matter of fact, Equation (1.6) fully encompasses the Ultimate limit states design philosophy. It meets the rationale of ancient builders and the “load testing method”, with the implicit belief that the concerned structure is safe as long as equilibrium under the design loads is possible complying with the resistance design values. This clearly underscores the importance of the concept of design values, which is essential to the relevance of such an approach.

1.6 Comments

The various explicit forms successively proposed, in the preceding Sections, for the governing principles of Ultimate limit states design in view of practical implementation, evidence the benefit of a statement that would retain the structure of Equation (1.1). The theory of Yield design can provide a basis for a general answer to this purpose. Its basic principles will now be recalled together with the most relevant results, before presenting its practical application to Ultimate limit states design.

2. Theory of Yield design: fundamental statements

2.1 Notations

The framework of classical continuum mechanics will be adopted for this general presentation of the theory as it allows easy transpositions to such particular cases as curvilinear media, plates,… and mixed structures. With the following notations:

Ω : volume of the considered system

$\partial\Omega$: boundary of Ω

$\underline{x}$: position vector of the current point in Ω

$\underline{\underline{\sigma}}$: Cauchy stress tensor field

$\underline{U}$: velocity field; $\underline{\hat{U}}$: virtual velocity field

$\underline{\underline{d}}$: strain rate field generated by $\underline{U}$; $\hat{\underline{\underline{d}}}$ generated by $\hat{\underline{U}}$

: jump of a quantity when crossing surface Σ following unit normal $\underline{n}$

“ . ” : symbol for a simple contracted product ($\underline{\underline{\sigma}}:\underline{n} = \sigma_{ij} n_j$)

“ : ” : symbol for a double contracted product ($\underline{\underline{\sigma}}:\underline{\underline{d}} = \sigma_{ij} d_{ji}$)

“ $\otimes$ ” : symbol for a tensor product.

Sign convention for stresses follows usual classical continuum mechanics practice: tensile stresses are counted positive.

2.2 Principle of virtual work. Loading parameters

The considered system is assumed to be subjected to a loading mode depending on a finite number n of scalar parameters, in such a way that the principle of virtual work can be written as:

$$\begin{aligned} \int_\Omega \underline{\underline{\sigma}}(\underline{x}):\hat{\underline{\underline{d}}}(\underline{x})\,\mathrm{d}\Omega + \int_\Sigma \left[\!\left[\hat{\underline{U}}(\underline{x})\right]\!\right].\underline{\underline{\sigma}}(\underline{x}).\underline{n}(\underline{x})\,\mathrm{d}\Sigma &= Q_j(\underline{\underline{\sigma}})\,\dot{q}_j(\hat{\underline{U}}) \\ &= \underline{Q}(\underline{\underline{\sigma}}).\underline{\dot{q}}(\hat{\underline{U}}) \end{aligned} \tag{2.1}$$

where

$$\underline{\underline{\sigma}} \mapsto \underline{Q}(\underline{\underline{\sigma}}) \in \mathbb{R}^n \tag{2.2}$$

$$\hat{\underline{U}} \mapsto \underline{\dot{q}}(\hat{\underline{U}}) \in \mathbb{R}^n \tag{2.3}$$

are linear functions.

Components Q_j of load vector $\underline{Q}$ are the loading parameters, with associated kinematic parameters $\dot{q}_j$, the components of $\underline{\dot{q}}$.

Let it be recalled that statically admissible (S.A.) stress fields are piecewise continuous and continuously differentiable and satisfy the equilibrium equations over Ω and the boundary conditions on $\partial\Omega$. Kinematically admissible (K.A.) velocity fields are piecewise continuous and continuously differentiable over Ω and comply with velocity boundary conditions on $\partial\Omega$. It is worth noting that virtual velocity jumps $\left[\!\left[\hat{\underline{U}}\right]\!\right]$ are not subject to any restriction.

2.3 Position of the problem

The system under consideration is geometrically defined, as already described in Section 2.1, by its volume Ω and boundary $\partial\Omega$, corresponding to its initial geometrical configuration when the exerted load is zero. The definition of the loading parameters through the statement of the principle of virtual work on this configuration (2.1) relies on the assumption that geometrical changes of the system will remain negligeable.

As already stated in Section 2.2, the system is subjected to a loading mode that depends on n parameters Q_j.

Its mechanical properties are defined by the data, at any point $\underline{x}$ of Ω, of the resistance characteristics of the constituent material (which may be heterogeneous) through a domain $G(\underline{x}) \in \mathbb{R}^6$ that defines the admissible stress states:

$$\forall \underline{x} \in \Omega,\ \underline{\underline{\sigma}}(\underline{x}) \in G(\underline{x}) \in \mathbb{R}^6 \tag{2.4}$$

with the following properties

$$\underline{\underline{\sigma}}(\underline{x}) = 0 \in G(\underline{x})$$
$$\underline{\underline{\sigma}}(\underline{x}) \text{ is star-shaped with respect to } 0\ ;$$

in addition, in practice, $G(\underline{x})$ is convex, a hypothesis that will be retained all along (although the theory can be established within a more general framework; cf., Salençon, 1983).

The question to be answered can now be stated as

Given a load $\underline{Q}$, is it possible to decide upon the stability or failure of the system subjected to this load?

2.4 Potentially safe loads. Extreme loads

The answer to this question proceeds from the following obvious necessary condition.

In order that the system sustain load $\underline{Q}$ under resistance condition (2.4) it is necessary that

$$\left.\begin{array}{c} \exists \underline{\underline{\sigma}} \text{ statically admissible with } \underline{Q} \\ \text{and abiding with (2.4)} \end{array}\right\} \tag{2.5}$$

Such a load $\underline{Q}$ that complies with (2.5) is said to be *potentially safe* and the system is *potentially stable* when subjected to this load.

The set generated by all potentially safe loads is denoted by K and defined through

$$\underline{Q} \in K \subset \mathbb{R}^n \Leftrightarrow \exists \underline{\underline{\sigma}} \left| \begin{array}{l} \text{S.A. with } \underline{Q} \\ \underline{\underline{\sigma}}(\underline{x}) \in G(\underline{x}), \forall \underline{x} \in \Omega \end{array}\right. \tag{2.6}$$

where field $\underline{\underline{\sigma}}$ is associated with $\underline{Q}$ through (2.2).

As a result of definition (2.6) and the linearity of (2.2), K is endowed with the following properties

$$\left.\begin{array}{l} \underline{Q} = 0 \in K \\ K \text{ is convex} \end{array}\right\} \tag{2.7}$$

Loads at the boundary of K are called *extreme loads* for the system. This recalls that, although any load in K is only *potentially* safe due to the fact that definition (2.5) is just a necessary condition for "stability", any load outside of K is definitely "*unsafe*": it cannot be sustained by the system (Figure 2).

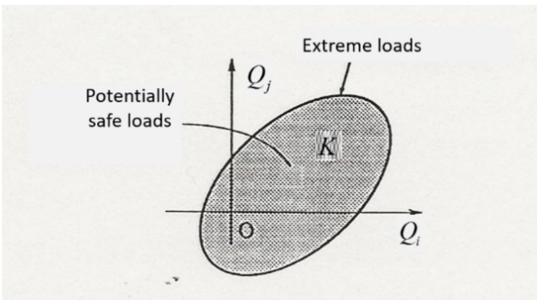

Figure 2. Potentially safe loads. Extreme loads

2.5 Comments

The answer given above to the question in Section 2.3 is only partial as shown by its including the term "potentially". This is a direct consequence of the fact that the only available data about the mechanical properties of the system for this analysis are the resistance of the constituent material; moreover, the question only refers to the given load $\underline{Q}$, without any information about the initial self-equilibrating stress field (when $\underline{Q} = 0$) or the loading history.

Thus, what may appear as a shortcoming of the outcome is also what makes it noteworthy. Domain K is independent of the initial self-equilibrating stress field, the loading history or the mechanical properties of the constituent materials abiding with (2.4), provided the assumption about negligeable geometry changes is satisfied. The fact that, in common practice, all or part of these information may either be missing or, in some cases, erroneous, enhances the interest of the analysis.

Checking the relevance of domain K and related concepts for practical applications amounts to checking to what extent potentially safe loads will be actually supported by the system.

A theoretical answer to this question was brought out in the case of a system made from an elastic and perfectly plastic material, governed by the principle of maximum plastic work (Brezis, 1973) (cf., Halphen and Salençon, 1987): any load within K will be actually supported. Such a textbook case is seldom encountered in common practice and the relevance of K must be assessed on the basis of experiments and the engineer experience. Particular attention shall be paid to the fact that the practical relevance of a Yield design analysis also depends on the physical compatibility of the deformations necessary for mobilising the resistance of the concerned elements in the system. In addition, the geometrical assumptions shall be validated.

2.6 Statical interior approach

As a straight-forward outcome of definition (2.6) we obtain a construction method of domain K: any stress field $\underline{\underline{\sigma}}$ that is statically admissible in the given loading mode and complies with (2.4) defines, through (2.2), a potentially safe load $\underline{Q}(\underline{\underline{\sigma}})$:

$$\left.\begin{array}{l}\underline{\underline{\sigma}} \text{ S.A. in the loading mode} \\ \underline{\underline{\sigma}}(\underline{x}) \in G(\underline{x}), \forall \underline{x} \in \Omega\end{array}\right| \xrightarrow{(2.2)} \underline{Q} \in K \qquad (2.8)$$

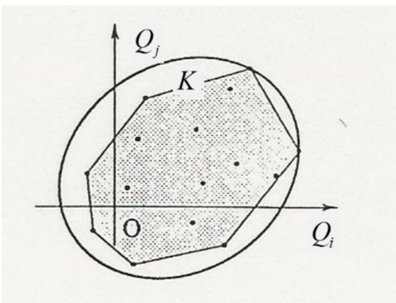

Figure 3. Statical interior approach

The convex hull of such loads $\underline{Q}$ provides an interior approach to K or a "lower bound" for the extreme loads (Figure 3).

An exact determination of K calls for a complete exploration of the set generated by the stress fields $\underline{\underline{\sigma}}$ in (2.8) (or a lot of luck!). Letting aside the case of simple curvilinear structures, common practice will settle for an interior approach of K chosen as a good balance between the simplicity of the means used and the quality of the results so-obtained. This highlights the usefulness of a complementary approach that would provide an "upper bound" for the extreme loads.

2.7 Fundamental statement of the kinematical approach

For any potentially safe load $\underline{Q}$, in view of definition (2.6) of domain K, Equation (2.1) of the principle of virtual work can be written with the stress field $\underline{\underline{\sigma}}$ associated with $\underline{Q}$ through (2.6) and any virtual velocity field $\underline{\hat{U}}$ kinematically admissible with the loading mode:

$$\begin{aligned}\int_{\Omega} \underline{\underline{\sigma}}(\underline{x}):\underline{\underline{\hat{d}}}(\underline{x})\,\mathrm{d}\Omega + \int_{\Sigma} [\![\underline{\hat{U}}(\underline{x})]\!] . \underline{\underline{\sigma}}(\underline{x}) . \underline{n}(\underline{x})\,\mathrm{d}\Sigma &= \underline{Q}(\underline{\underline{\sigma}}) . \underline{\dot{q}}(\underline{\hat{U}}) \\ &= \mathcal{P}_{\mathrm{e}}(\underline{Q}, \underline{\hat{U}})\end{aligned} \tag{2.9}$$

Let functions "π" be defined as follows

$$\pi(\underline{x}, \underline{\underline{\hat{d}}}(\underline{x})) = \mathrm{Sup}\left\{ \underline{\underline{\sigma}}'(\underline{x}):\underline{\underline{\hat{d}}}(\underline{x}) \,\middle|\, \underline{\underline{\sigma}}'(\underline{x}) \in G(\underline{x}) \right\} \tag{2.10}$$

$$\pi(\underline{x}, \underline{n}(\underline{x}), [\![\underline{\hat{U}}(\underline{x})]\!]) = \mathrm{Sup}\left\{ [\![\underline{\hat{U}}(\underline{x})]\!] . \underline{\underline{\sigma}}'(\underline{x}) . \underline{n}(\underline{x}) \,\middle|\, \underline{\underline{\sigma}}'(\underline{x}) \in G(\underline{x}) \right\} \tag{2.11}$$

Since stress field $\underline{\underline{\sigma}}$ in (2.9) complies with (2.4), it follows from (2.9)-(2.11) that

$$\begin{aligned}&\forall \underline{Q} \in K,\ \forall \underline{\hat{U}} \text{ K.A.},\\ &\mathcal{P}_{\mathrm{e}}(\underline{Q}, \underline{\hat{U}}) = \underline{Q} . \underline{\dot{q}}(\underline{\hat{U}}) \leq \int_{\Omega} \pi(\underline{x}, \underline{\underline{\hat{d}}}(\underline{x}))\,\mathrm{d}\Omega + \int_{\Sigma} \pi(\underline{x}, \underline{n}(\underline{x}), [\![\underline{\hat{U}}(\underline{x})]\!])\,\mathrm{d}\Sigma\end{aligned} \tag{2.12}$$

Denoting by $\mathcal{P}_{\mathrm{mr}}(\underline{\hat{U}})$ the maximum resisting work in field $\underline{\hat{U}}$ defined as

$$\mathcal{P}_{\mathrm{mr}}(\underline{\hat{U}}) = \int_{\Omega} \pi(\underline{x}, \underline{\underline{\hat{d}}}(\underline{x}))\,\mathrm{d}\Omega + \int_{\Sigma} \pi(\underline{x}, \underline{n}(\underline{x}), [\![\underline{\hat{U}}(\underline{x})]\!])\,\mathrm{d}\Sigma\,, \tag{2.13}$$

Equation (2.12) can be written, in a compact form, as

$$\left.\begin{aligned}&\forall \underline{\hat{U}} \text{ K.A.},\\ &K \subset \left\{ \mathcal{P}_{\mathrm{e}}(\underline{Q}, \underline{\hat{U}}) \leq \mathcal{P}_{\mathrm{mr}}(\underline{\hat{U}}) \right\}\end{aligned}\right\} \tag{2.14}$$

2.8 π functions

2.8.1 The mechanical significance of π functions

The compact statement above is the result of the introduction of π functions defined through (2.10)-(2.11), whose mechanical significance shall be investigated.

Equation (2.10) shows that, given $\underline{\underline{\hat{d}}}(\underline{x})$, a virtual strain rate at point $\underline{x}$, $\pi(\underline{x}, \underline{\underline{\hat{d}}}(\underline{x}))$ is the maximum value of the work that can be produced in $\underline{\underline{\hat{d}}}(\underline{x})$ by any stress field $\underline{\underline{\sigma}}'(\underline{x})$ that complies with the condition of resistance (2.4) of the constituent material. Hence, $\pi(\underline{x}, \underline{\underline{\hat{d}}}(\underline{x}))$ comes out as the volume density of maximum resisting work in virtual strain rate $\underline{\underline{\hat{d}}}(\underline{x})$ under the resistance condition defined by $G(\underline{x})$.

From a mathematical viewpoint, $\pi(\underline{x},\bullet)$ is the support function of convex $G(\underline{x})$ and it is known, from simple convex analysis, that the data of this function defines $G(\underline{x})$ just as well as a criterion, which means that $\pi(\underline{x},\bullet)$ contains, in a dual form, all information embedded in $G(\underline{x})$.

The justification of the terminology "resisting" comes from Equation, (2.1) where the work by external forces is balanced by the resisting work defined as the opposite of the work by internal forces (Germain, 1986).

All that has just been said about function $\pi(\underline{x},\bullet)$ is valid for function $\pi(\underline{x},\underline{n}(\underline{x}),\bullet)$ defined by (2.11) since, as it can easily be verified:

$$\pi(\underline{x},\underline{n}(\underline{x}),[\![\underline{\hat{U}}(\underline{x})]\!])=\frac{1}{2}\pi(\underline{x},\underline{n}(\underline{x})\otimes[\![\underline{\hat{U}}(\underline{x})]\!]+[\![\underline{\hat{U}}(\underline{x})]\!]\otimes\underline{n}(\underline{x})) \tag{2.15}$$

It is worth noting that the fundamental idea of this kinematical approach, which lies in defining $G(\underline{x})$ in a dual way by means of the π functions, has been brought forward by Prager (1955) within the framework of the theory of plasticity.

2.8.2 The computation of π functions

In view of (2.15), only functions $\pi(\underline{x},\underline{\underline{\hat{d}}}(\underline{x}))$ will be considered here. The two typical cases to be dealt with, when computing this function, are schematically presented in Figure 4.

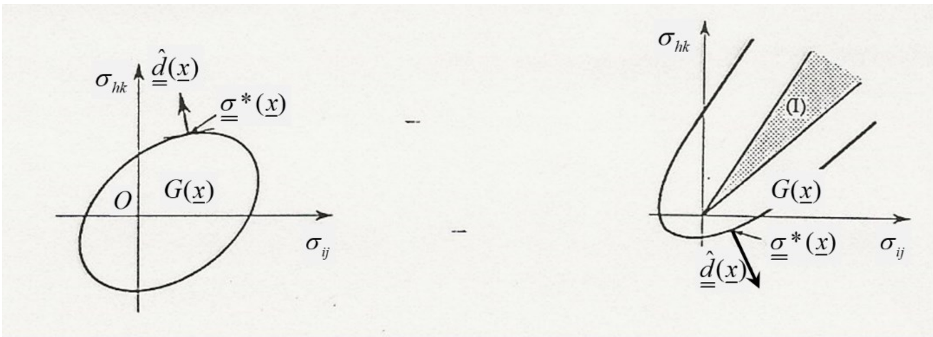

Figure 4. The computation of $\pi(\underline{x},\underline{\underline{\hat{d}}}(\underline{x}))$

1° If convex $G(\underline{x})$ is bounded in all directions in $\mathbb{R}^6$ we get

$$\pi(\underline{x},\underline{\underline{\hat{d}}}(\underline{x}))=\underline{\underline{\sigma}}^*(\underline{x}):\underline{\underline{\hat{d}}}(\underline{x}), \tag{2.16}$$

where $\underline{\underline{\sigma}}^*(\underline{x})$ denotes one of possibly several stress states at the boundary of $G(\underline{x})$ where $\underline{\underline{\hat{d}}}(\underline{x})$ is colinear with an outward normal (same orientation). In this case, $\pi(\underline{x},\underline{\underline{\hat{d}}}(\underline{x}))$ takes a finite value whatever $\underline{\underline{\hat{d}}}(\underline{x})$.

2° If convex $G(\underline{x})$ is not bounded in all directions, let (I) denote the convex cone of its directions at infinity in $\mathbb{R}^6$. Then,

- If $\underline{\underline{\hat{d}}}(\underline{x})$ lies within the convex cone orthogonal to (I), $\pi(\underline{x},\underline{\underline{\hat{d}}}(\underline{x}))$ takes the finite value

$$\pi(\underline{x},\underline{\underline{\hat{d}}}(\underline{x}))=\underline{\underline{\sigma}}^*(\underline{x}):\underline{\underline{\hat{d}}}(\underline{x}), \tag{2.17}$$

where $\underline{\underline{\sigma}}^*(\underline{x})$ is defined as in the preceding case;

- If $\hat{\underline{\underline{d}}}(\underline{x})$ does not belong to this cone, $\pi(\underline{x},\hat{\underline{\underline{d}}}(\underline{x}))$ is infinite

$$\pi(\underline{x},\hat{\underline{\underline{d}}}(\underline{x})) = +\infty \tag{2.18}$$

In the mechanics of three-dimensional continuous media, the domains of resistance that are usually implemented belong to the second case considered above (e.g., Tresca's, Mises', Coulomb's, Drucker-Prager's criteria…): hence, the associated π functions take finite or infinite values depending on the considered virtual strain rate. The case of a π function that would take an infinite value whatever $\hat{\underline{\underline{d}}}(\underline{x})$ would describe a material with an infinite resistance whatever the action it is subjected to.

It turns out that, under certain loading conditions of a three-dimensional continuum, such as plane stress for instance, and in the case of one- or two-dimensional continua (beams, plates), the domains of resistance are bounded in all directions (1^{st} case in Figure 4).

A large but non exhaustive list of resistance criteria and associated π functions is available in (Salençon, 1983).

From a mathematical viewpoint, π functions are endowed with the following properties:

$\pi(\underline{x},\bullet)$ is non-negative

$\pi(\underline{x},\bullet)$ is positively homogeneous with degree 1

$\pi(\underline{x},\bullet)$ is convex.

2.9 Kinematical approach

2.9.1 Exterior approach

Statement (2.14) is the basis of the kinematical exterior approach of domain K.

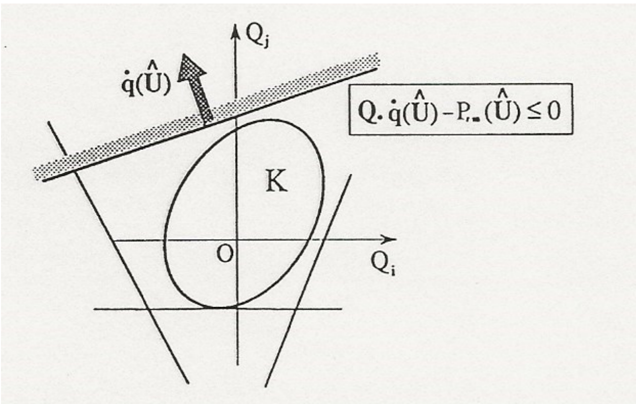

Figure 5. Kinematical exterior approach

For any kinematically admissible virtual velocity field $\hat{\underline{U}}$, the maximum resisting work $\mathcal{P}_{mr}(\hat{\underline{U}})$ can be computed from the available expressions for the π functions, while the work by external forces is $\mathcal{P}_e(\underline{Q},\hat{\underline{U}}) = \underline{Q}.\dot{\underline{q}}(\hat{\underline{U}})$, a linear form in $\underline{Q}$ where $\dot{\underline{q}}(\hat{\underline{U}})$ is known from (2.3). It follows that domain K is included in the half-space defined by Equation (2.14) in $\mathbb{R}^n$. Since this implementation can be repeated with various fields $\hat{\underline{U}}$, an exterior approach of domain K can rapidly be obtained, as shown in Figure 5, together with an "upper bound" for the extreme loads.

In addition, it was proven, at the price of a few complementary mathematical assumptions (Nayroles, 1970; Frémond & Friaà, 1978; Friaà, 1979) that (2.14) is definitely the exact dual definition of K , i.e., K can be generated through the implementation of (2.14) over all kinematically admissible virtual velocity fields.

2.9.2 Relevant velocity fields

Implementing Equation (2.14) only results in a non-trivial outcome if

- The work by external forces is non-zero, $\mathcal{P}_{\mathrm{e}}(\underline{Q},\underline{\hat{U}}) \neq 0$
- The maximum resisting work is finite, $\mathcal{P}_{\mathrm{mr}}(\underline{\hat{U}}) < +\infty$.

The first of these two conditions does not call for any comment, since it is just common sense.

The second implies that $\underline{\hat{U}}$ must be chosen in such a way that $\pi(\underline{x},\underline{\underline{\hat{d}}}(\underline{x}))$ and $\pi(\underline{x},\underline{n}(\underline{x}),[\![\underline{\hat{U}}(\underline{x})]\!])$ be finite. Such virtual velocity fields will be called *relevant*. It is worth underscoring that this concept is uniquely related to the dual form of the condition of resistance (2.4) and is not, by any means (other than mathematical similarity), related to a plastic flow rule or any other hypothesis as to the behaviour of the constituent material when the condition of resistance is saturated.

2.10 Final remarks

To conclude this brief presentation of the theory of Yield design it is worth recalling that it only relies on a necessary condition for the stability of the considered system that reads as

$$\textit{Compatibility}\ \text{between}\ \begin{cases} \textit{equilibrium}\ \text{(system)} \\ \textit{resistance}\ \text{(material)} \end{cases} \tag{2.19}$$

without any other data.

The statical approach, a logical outcome of this condition, was implemented by Hill (1954, 1966).

The kinematical approach, as expressed by (2.14), is derived from the dual formulation of the statical approach and involves *virtual* kinematically admissible velocity fields, without any reference to complementary assumptions about the behaviour of the constituent material.

3. Formulation and justification of Ultimate limit states design through the theory of Yield design

3.1 Identifying the issues

3.1.1 Basic principles are identical

Bringing together the basic principles of Ultimate limit states design (ULSD) introduced in Section 1.1 and the basic principles of Yield design analysis recalled as (2.19) in Section 2.10 evidences that both analyses only rely on the concepts of *equilibrium* and *resistance* that shall be compatible with each other.

In order to render the problems identical, the parameters involved in each of them must be precisely matched.

3.1.2 The loads and their design values

As already stated in Section 1.2, the actions exerted on a structure designed within the ULSD framework consist of one or several load combinations, whose values are given and factored by means of multipliers to define their design values.

Matching with theory of Yield design can simply be obtained assigning a loading parameter Q_j to each of the loads.

The issue then, consists in considering the loads $\underline{Q}(S_{\mathrm{d}})$ corresponding to the various prescribed combinations of loads with their appropriate partial safety factors, and locating them in relation to the domain $K(R_{\mathrm{d}})$ defined in following Section.

Since the partial safety factor to be applied to a load takes different values depending on the load being favourable or unfavourable to equilibrium, it is necessary, for each load combination, to check the situation of the all the loads $\underline{Q}(S_{\mathrm{d}})$ obtained when considering all permutations of the minimum and maximum values of the partial safety factors. This is schematically presented in Figure 6, where it appears that, for each load combination, the process amounts to checking that the summits of an hypercube in $\mathbb{R}^n$ lie actually within $K(R_{\mathrm{d}})$.

Here is a point, where convexity of domain $K(R_{\mathrm{d}})$ in the loading parameters space, which is proven from the theory of Yield design (Section 3.1.3), comes out as an essential argument since it ensures that the hypercube will entirely be included in $K(R_{\mathrm{d}})$.

3.1.3 The resistances and their design values

The resistance characteristics of the constituent materials of the structure designed within the ULSD framework are defined by means of resistance parameters, denoted symbolically by R, whose design values are R_{d} (Section 1.3).

Matching with the theory of Yield design is simply achieved by making the resistance parameters appear explicitly in the condition of resistance (2.4):

$$\forall \underline{x} \in \Omega,\ \underline{\underline{\sigma}}(\underline{x}) \in G(R;\underline{x}) \tag{3.1}$$

where $G(R;\underline{x})$ is defined through the resistance criterion introduced in Section 1.4:

$$f\big(R;\underline{x},\underline{\underline{\sigma}}(\underline{x})\big) \leq 0 \Leftrightarrow \underline{\underline{\sigma}}(\underline{x}) \in G(R;\underline{x}) \tag{3.2}$$

It is assumed that, whatever the resistance parameters R, $G(R;\underline{x})$ retains the same properties as listed in Section 2.3, among which *convexity*[2].

From the theory of Yield design, it is then possible to determine domain $K(R)$ in the loading parameters space defined here above, that complies with (2.7).

The design of the structure within the ULSD framework refers to the domain $K(R_{\mathrm{d}})$ that corresponds to the design values of the resistance parameters.

3.2 A statical theoretical formulation of Ultimate limit states design

Figure 6 schematically sums up the results of the preceding Sections as to matching Ultimate limit states design with the Yield design theory. It consists in checking that the various loads $\underline{Q}(S_{\mathrm{d}})$ stay within $K(R_{\mathrm{d}})$

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[2] An example of a resistance criterion, whose convexity is not proven for all practical values of the resistance parameters can be found in (Salençon & Tristan-López, 1983).

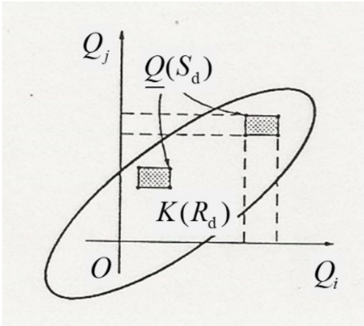

Figure 6. Statical formulation of ULSD

With $\mathbb{S}(S_\mathrm{d})$ denoting the set of all statically admissible stress fields, in the same way as in Section 1.4.4, in view of definitions (2.6) for K and (3.2) for $G(R;\underline{x})$, the process results in checking, for any value of S_d, if there exist at least one stress field $\underline{\underline{\sigma}}$ such that

$$\left.\begin{array}{l} \underline{\underline{\sigma}} \in \mathbb{S}(S_\mathrm{d}) \\ f\big(R_\mathrm{d}\,;\underline{x},\underline{\underline{\sigma}}(\underline{x})\big) \leq 0,\ \forall \underline{x} \in \Omega \end{array}\right\} \tag{3.3}$$

i.e.,

$$\operatorname*{Inf}_{\underline{\underline{\sigma}} \in \mathbb{S}(S_\mathrm{d})} \ \operatorname*{Sup}_{\underline{x} \in \Omega} f\big(R_\mathrm{d}\,;\underline{x},\underline{\underline{\sigma}}(\underline{x})\big) \leq 0 \ . \tag{3.4}$$

This is precisely the statement (1.6) that had been proposed in Section 1.4.4 as an explicit form for the fundamental statement (1.1) of ULSD. in the case of statically indeterminate structures and systems.

Equations (3.3) and (3.4) provide the statical theoretical formulation of Ultimate limit states design.

3.3 About partial safety factors on loads and resistances

Various considerations are usually put forward to justify, in practice, the introduction of partial safety coefficients on loads and resistances to determine their design values. Some of them refer to the different stochastic characters of loads and resistances. Matching ULSD with the Yield design theory brings out, as one more argument, the fact, already commented in Sections 1.5 and 2.5, that the answer to the question about the stability of the concerned structure is only partial.

3.4 A kinematical theoretical formulation of Ultimate limit states design

3.4.1 Statement

For simplicity's sake, it is assumed that the complementary mathematical hypotheses necessary for proving that Equation (2.14) is the exact dual definition of K (Section 2.9.1) are satisfied, so that

$$K = \bigcap_{\underline{\hat{U}}\ \text{K.A.}} \left\{ \mathcal{P}_\mathrm{e}(\underline{Q},\underline{\hat{U}}) \leq \mathcal{P}_\mathrm{mr}(\underline{\hat{U}}) \right\} \ . \tag{3.5}$$

It is now necessary to make the resistance parameters R, which define $G(R;\underline{x})$ and, consequently, the π functions, appear explicitly. Hence

$$\mathcal{P}_\mathrm{mr}(R;\underline{\hat{U}}) = \int_\Omega \pi(R;\underline{x},\underline{\underline{\hat{d}}}(\underline{x}))\,\mathrm{d}\Omega + \int_\Sigma \pi(R;\underline{x},\underline{n}(\underline{x}),\big[\!\big[\underline{\hat{U}}(\underline{x})\big]\!\big])\,\mathrm{d}\Sigma \tag{3.6}$$

Then, from (3.5), the dual definition of $K(R_\mathrm{d})$, we derive the dual kinematical form equivalent to (3.3) or (3.4), which reads, for any load $\underline{Q}(S_\mathrm{d})$

$$\left.\begin{array}{l}\forall \underline{\hat{U}} \text{ K.A.,} \\ \mathcal{P}_{\mathrm{e}}(S_{\mathrm{d}}, \underline{\hat{U}}) \leq \mathcal{P}_{\mathrm{mr}}(R_{\mathrm{d}}; \underline{\hat{U}})\end{array}\right\} \tag{3.7}$$

with, as a definition,

$$\mathcal{P}_{\mathrm{e}}(S_{\mathrm{d}}, \underline{\hat{U}}) = \mathcal{P}_{\mathrm{e}}(\underline{Q}(S_{\mathrm{d}}), \underline{\hat{U}}) . \tag{3.8}$$

This is the *kinematical theoretical formulation* of Ultimate limit states design.

3.4.2 Comments

The benefit of this formulation is obvious when one refers to the comments in Section 1.5, since it results in an expression that retrieves the structure of Equation (1.1).

Here, S_{d} and R_{d} denote the *design values* of the loads and resistance parameters, whose effects are separated from each other and appear, for each of them, in the form of a scalar quantity, with the dimension of a work, that is computed in a kinematically admissible virtual velocity field. Equation (3.7) must be satisfied *for all such fields* $\underline{\hat{U}}$.

For each field $\underline{\hat{U}}$, it is easy to identify "unfavourable" and "favourable" loads in relation to equilibrium, which makes it possible to affect them with the appropriate coefficient from the start. Indeed, it is sufficient to check the sign of the contribution of the concerned load to $\mathcal{P}_{\mathrm{e}}(S_{\mathrm{d}}, \underline{\hat{U}})$: if is positive, the load is "unfavourable", if it is negative, it is "favourable".

3.5 Practical implementation of the kinematical formulation of Ultimate limit states design

It thus appears that Equation (3.7) provides a precise answer to the question that was brought forward in the first Section of this contribution.

Full exploitation of this equation would imply an exhaustive exploration of the set of relevant kinematically admissible $\underline{\hat{U}}$ (Section 2.9.2). Common practice usually settles for a partial exploitation, more or less sophisticated, of this equation exploring a subset of relevant kinematically admissible velocity fields $\underline{\hat{U}}$.

To guard against the bias due to such a partial implementation, the idea came to introduce a *method coefficient*, Γ_{S} superior to 1, that depends on the method used, i.e., on the subset $\mathbb{C}_{\mathrm{M}}$ of relevant kinematically admissible velocity fields $\underline{\hat{U}}$ that is explored, in the form

$$\left.\begin{array}{l}\forall \underline{\hat{U}} \in \mathbb{C}_{\mathrm{M}} \\ \Gamma_{\mathrm{S}} \mathcal{P}_{\mathrm{e}}(S_{\mathrm{d}}, \underline{\hat{U}}) \leq \mathcal{P}_{\mathrm{mr}}(R_{\mathrm{d}}; \underline{\hat{U}})\end{array}\right\} \tag{3.9}$$

This point can be schematically illustrated with an example in the field of soil mechanics.

For the bearing capacity of a strip footing with width B on a purely cohesive soil with cohesion C , full exploitation of Equation (3.7) classically determines the exact value

$$F = (\pi + 2) B C , \tag{3.10}$$

which can be obtained, for instance, using Prandtl's virtual velocity field (Figure 7).

A partial implementation of Equation (3.7), e.g., considering only relevant kinematically admissible velocity fields of the slip-circle type, results in the following estimate

$$F = 5.53BC\,. \tag{3.11}$$

Comparing these two results would naturally suggest the introduction of a method coefficient about 1.08 (or more).

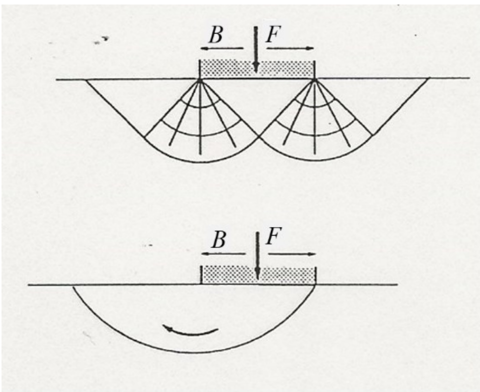

Figure 7. bearing capacity of a strip footing:
Prandtl's velocity field; slip-circle velocity field

Other considerations are usually brought forward in favour of the introduction of a method coefficient, which are object of controversy, while this simple example clearly evidences the necessity of such a coefficient as soon as Equation (3.7) is not implemented with all relevant kinematically admissible velocity fields.

It may be noted that, "by construction", Equation (3.7), as a kinematical formulation, does not refer to any specific class of kinematically admissible velocity fields $\underline{\hat{U}}$, so that it can be unambiguously implemented on velocity fields as different as those in Figure 7, and makes it possible to compare their "merits". In this sense, it appears as the complete answer to the initial question. It differs from other (more or less happy) attempts, which refer to specific classes of velocity fields or "failure modes".

4. Conclusion

The Theory of yield design stands as the basis of Ultimate limit states design.

Its statical approach allows a precise formulation of the issue while providing a theoretical justification for various current practices.

Its kinematical approach, which is derived by means of the principle of virtual work, provides a precise and complete formulation for the fundamental equation of Ultimate limit states design. The effects of loads and resistances are unambiguously separated, in the virtual work by external forces, on the one side, and in the maximum resisting work on the other side

$$\left.\begin{array}{l} \forall \underline{\hat{U}} \text{ K.A.,} \\ \mathcal{P}_{\mathrm{e}}(S_{\mathrm{d}}, \underline{\hat{U}}) \leq \mathcal{P}_{\mathrm{mr}}(R_{\mathrm{d}}; \underline{\hat{U}}) \end{array}\right\} \tag{4.1}$$

Partial implementations of this Equation on restricted sets of kinematically admissible velocity fields $\underline{\hat{U}}$ is one justification for the introduction of a method coefficient.

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