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# EVALUATION OF GLOBAL BEARING CAPACITIES OF STRUCTURES

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## **YIELD DESIGN: A SURVEY OF THE THEORY**

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### **ABSTRACT**

The theory of Yield Design is based upon the obvious necessary condition for the stability of a structure that the equilibrium of that structure and the resistance of its constituents should be compatible. The static approach of the yield design theory proceeds directly from this condition, leading to lower estimates of the extreme loads. The kinematic approach is derived by dualizing the static approach through the principle virtual work, thus ensuring full mechanical consistency. The treatment of a classical example illustrates these arguments. Present and possibly future domains of practical applications of the theory are reviewed, including the full adequacy between the Yield Design Theory and the Ultimate Limit State Design concept of safety.

### 1.- SOME HISTORICAL LANDMARKS

The concept of yield (or limit) design is most popular among civil engineers who have made an extensive use of it for centuries when designing structures or earthworks. As a matter of fact, one usually refers to Galileo's study [1] of the cantilever beam as the first written explication of the concept : the maximum load the beam may withstand is derived from the only knowledge of the strength of its constituents, namely the fibers (Fig.1).



*Figure 1 - Galileo's analysis of the cantilever beam.*

Coulomb's celebrated memoir [2] appears then as the reference where such problems as the compression of a column, the stability of a retaining wall or of a masonry arch, etc., are considered from the same point of view combining the equilibrium of the structure and the resistance of its constituents. Papers by Heyman [3-7] and by Delbecq [8, 9] may help getting a comprehensive view of Coulomb's analysis and of the subsequent works (e.g. Méry [10], Durand-Claye [11, 12], ...) concerning masonry works.

As regards soil mechanics, the original Coulomb analysis has been followed by others in the same "spirit" : the general equilibrium, in terms of resulting moments and resulting forces, of a bulk of soil, defined in the considered earthwork by one or a few parameters, is checked under the condition that the soil cannot withstand stresses outside its strength criterion usually defined as a Coulomb criterion. Among many, one must quote the Culmann method [13], and Fellenius' famous analysis for slope stability where the bulks of soil whose equilibrium is checked are limited by circular lines. When considering such circular lines in the case of a frictional soil, the reasoning can

only be carried out by introducing complementary assumptions as in the slices method for instance [14-17]. Rendulic [18] introduced the use of logspirals instead of circular lines in that case, which removes the difficulty and preserves the full significance of the analysis as it will be shown later on. Analogous analyses have been performed in order to investigate the stability of surface foundations.

For what concerns the stability of surface foundations and the determination of their bearing capacity, the limit equilibrium methods (as classified by Chen [19] for instance) have also been used, especially in the case of plane problems : the equations were first established by Massau [20] and Kötter [21, 22], and important developments may be found in the famous books by Sokolovski [23-25] and Berezancev [26]. Since the limit equilibrium equations for plane problems are homologous to the equations appearing in the study of plane strain problems for perfectly plastic materials obeying Tresca's yield criterion with associated flow rule [27], they are often associated with the concept of plasticity : in other words they are improperly said to rely on the assumption of a rigid perfectly plastic soil obeying Coulomb's criterion with associated flow rule. As a matter of fact, it should be clear to everybody that limit equilibrium methods are but another technique to perform the reasoning of the yield design theory, and that they do not require any more assumption about the constitutive law of the considered soil than the data of its strength criterion.

Another classical domain of application for the yield design theory concerns the analysis of metallic plates and reinforced concrete slabs, with the use of Johansen's criterion and yield line theory [28, 29] and other methods, as explained for instance in Save and Massonnet [30].

Recently the combination of the homogenization theory with the yield design theory has opened the access to new types of investigations and has already led to interesting results such as the determination of the "macroscopic" homogenized strength criterion of a composite based upon the data of the strength criteria of its constituent materials, including the interfaces between them, which is to be introduced in the homogenized yield design analysis of a structure made of that composite material. From the practical point of view the domain of relevance ranges from mechanical engineering with, for instance, the "classical" long fiber/epoxy matrix composites [31, 32] to civil engineering where soils reinforced by inclusions such as nails,

geotextiles and geogrids, or metallic strips, etc., are now being extensively used [33-35].

The purpose of the presentation of the yield design theory to be given hereafter is to cover those different types of applications, and others which have not been listed to make short. Starting from the simple example of a structure, in order to introduce and discuss the first fundamental concepts, it will then be developed within the framework of classical continuum mechanics with the help of a few basic notions in convex analysis.

Through its generality it is hoped to show the unity between existing methods in the various domains of applications of the yield design theory, together with their efficiency, and to arouse the interest for devising new developments which may take advantage of the everyday increasing possibilities offered by numerical techniques.

## 2.- AN INTRODUCTION TO THE YIELD DESIGN APPROACH.

### 2.1.- Definition of the problem under consideration

The square truss ABCD shown in Fig. 2 is made of six bars with pivot hinges in A, B, C and D, and no connection between bars AC and BD. This structure is submitted to an external force acting on joint C, in the direction of AC with intensity  $Q$ , while joint A is fixed. No other force is applied to the truss and  $Q$  appears naturally as the load parameter for the problem.

The internal efforts in the bars obviously reduce to tensile or compressive forces, namely  $N_1, N_2, \dots, N_6$  in bars AB, BC, CB, DA, AC and BD respectively.

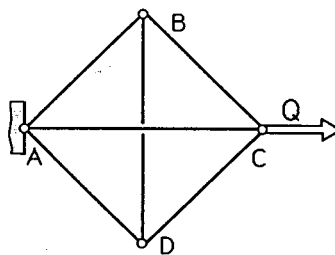


Figure 2 - One parameter loading of a simple structure.

It is assumed that the six bars of the truss exhibit the same resistance in tension and in compression, equal to  $L$ , so that condition (2.1) has to be satisfied :

$$|N_i| \leq L, \quad \forall i = 1, 2, \dots, 6. \quad (2.1)$$

The question to be answered is to determine the maximum and minimum values of parameter  $Q$  the structure can withstand under the strength conditions (2.1) imposed to the bars.

### 2.2.- Compatibility between equilibrium and resistance.

The equilibrium equations for the structure in Figure 2 are written

$$\begin{aligned} N_1 &= N_2 = N_3 = N_4, \\ N_5 + N_1\sqrt{2} &= Q, \\ N_6 + N_1\sqrt{2} &= 0. \end{aligned} \quad (2.2)$$

A necessary condition for that structure to withstand load  $Q$  under the strength conditions (2.1) is therefore that Eq. (2.2) be mathematically compatible with Ineq. (2.1). It follows that the intensity  $Q$  of the applied force should not exceed the conditions :

$$|Q| \leq 2L \quad (2.3)$$

This leads to the following statements :

- if the applied load  $Q$  is such that  $|Q| > 2L$ , it is sure that the structure cannot be in equilibrium together with the strength conditions being satisfied at the same time ;
- if the applied load  $Q$  is such that  $|Q| \leq 2L$ , it is possible that the structure be in equilibrium and the strength conditions be satisfied at the same time.

Briefly speaking, introducing the termes "stable" and "unstable" to characterize those circumstances :

$$\begin{aligned} |Q| > 2L : & \text{ the structure cannot be stable under } Q, \\ & \text{ it is } \textit{surely unstable} \text{ under } Q ; \\ |Q| \leq 2L : & \text{ the structure may be stable under } Q, \\ & \text{ it is } \textit{potentially stable} \text{ under } Q. \end{aligned} \quad (2.4)$$

The loads  $Q^+ = 2L$  and  $Q^- = -2L$  will be called the extreme loads of the structure in the loading mode defined in Fig. 2.

### 2.3.- Comments.

Statement (2.4) deserves many comments, most of which will remain relevant in the general case.

#### 2.3.1.

The analysis has been performed on the given geometry of the structure : no geometry changes are taken into account.

#### 2.3.2.

The only data required for the analysis given in Section (2.2) are the strength conditions imposed on the bars. It follows that results (2.4) hold whatever the initial internal forces in the structure for  $Q = 0$ , whatever the loading path, whatever the constitutive equations for the bars provided they are consistent with the strength conditions (2.1) and with the preceding comment regarding the geometry where the equations are written.

#### 2.3.3.

Statement (2.4) is but a partial answer to the original question asked in Section (2.1) since it only offers a guarantee of instability (!) when  $|Q| > 2L$  and a presumption of stability when  $|Q| \leq 2L$ .

It should be understood that such a conclusion is the maximum one can logically derive starting from the only available data (2.1) on the resistance of the constituent elements of the structure imposed as a constraint. In order to be able to assert the stability of the structure under a given load  $Q$  it would be necessary that complementary information regarding

- the mechanical behaviour of the constituent elements, through their constitutive equations for instance,
- the initial state of self-equilibrated interior forces,
- the loading history followed to reach the actual load  $Q$  from the unloaded initial state be available.

As a matter of fact it can be easily perceived that the condition for the structure to withstand an extreme load is that the strength capacities required to equilibrate that load can be actually mobilized simultaneously in the concerned elements, at the end of the concerned loading path, starting from the given initial state of self equilibrated interior forces, and with the given constitutive equations.



Fig. 4 recalls an example from [36] where the response of the structure introduced in Fig.2 is studied for a monotonous increasing loading path, starting from the zero self equilibrated initial state of interior forces, under two different assumptions for the constitutive equations of the elements (Fig.3) :

- a) the bars are elastic and perfectly plastic in tension and in compression,
- b) the bars are elastic and perfectly plastic in compression, elastic and brittle in tension.

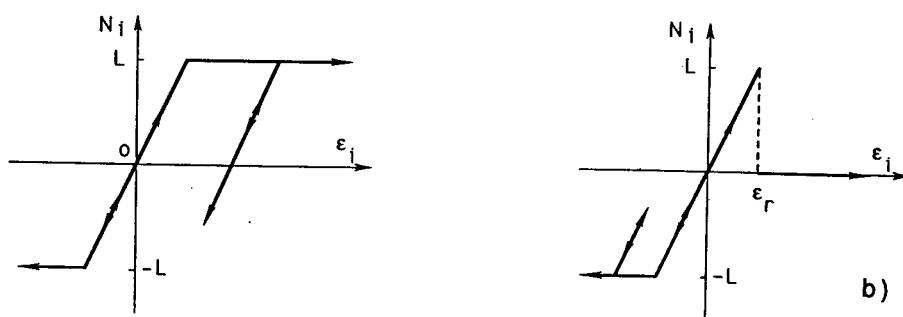


Figure 3 - Elastic perfectly plastic (a), and elastic brittle/perfectly plastic (b) behaviours for the bars in Fig. 2.

Although the latter assumption might seem somewhat schematic, the results shown in Fig. 4 point out that in the first case the structure will actually withstand any load between  $Q^-$  and  $Q^+$ , whereas in the second case the structure will break when  $Q$  reaches the level  $Q = L\sqrt{2}$ .

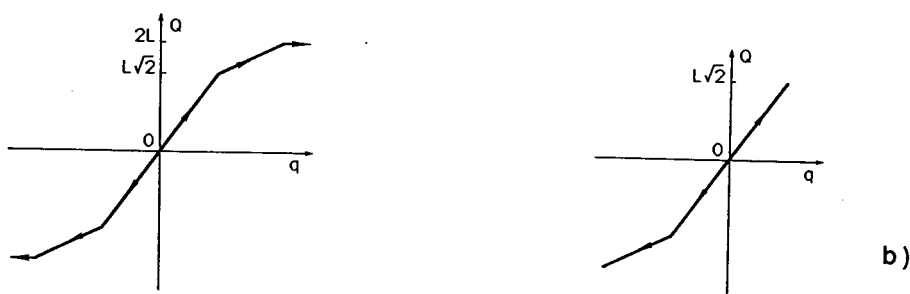


Figure 4 - Response of the truss in Fig. 2 with the constitutive equations shown in Fig. 3 for the bars.

The preeminent role of ductility in the mechanical behaviour of the bars is quite apparent here. Generally speaking such ideal behaviours as shown in Fig. 3 will scarcely be encountered when dealing with practical applications. Nevertheless it is clear that the idea expressed by Jewell [37] that the deformations necessary to fully mobilize the required resistance of the elements, in the structure, along the considered loading path, must be "compatible" with each other must be retained.

### 3.- PRINCIPLES OF THE THEORY OF YIELD DESIGN : STATIC APPROACH

#### 3.1.- *Notations*

The theory will be presented within the frame of classical continuum mechanics.

The following notations will be used :

- $\Omega$  : volume of the system under consideration ;
- $\partial\Omega$  : boundary of  $\Omega$  ;
- $\mathbf{x}$  : vector defining the position of the current point in  $\Omega$  ;
- $\boldsymbol{\sigma}$  : Cauchy stress field ;
- $\mathbf{U}$  : velocity field ;  $\mathbf{U}$  : virtual velocity field ;
- $\mathbf{d}$  : strain rate field derived from  $\mathbf{U}$  , ( $\mathbf{d}$  from  $\mathbf{U}$ ) ;
- $[ ]$  : symbol for the jump accross surface  $\Sigma$  following unit normal  $\mathbf{n}$  ;
- $\cdot$  : symbol for the contracted product (e.g.  $\boldsymbol{\sigma} \cdot \mathbf{n} = \sigma_{ij} n_j$  and  $\mathbf{T} \cdot \mathbf{U} = T_i U_i$  <sup>(1)</sup>) ;
- $:$  : symbol for the twice contracted product (e.g.  $\boldsymbol{\sigma} : \mathbf{d} = \sigma_{ij} d_{ji}$ ).

#### 3.2.- *Principle of virtual work (powers). Loading parameters.*

The theory of yield design is most conveniently formulated in the case when the loading mode of the system depends on a finite number of loading parameters, which is always the case in practical applications. The definition of such loading parameters is given in [36]. It may be shortly expressed as follows, through the principle of virtual work in statics.

(<sup>1</sup>) Orthonormal cartesian coordinates.

*For any stress field  $\underline{\sigma}$ , statically admissible for the problem in the loading mode, for any virtual velocity field  $\hat{\mathbf{U}}$ , kinematically admissible for the problem in the loading mode, the following equation holds*

$$\int_{\Omega} \underline{\sigma}(\mathbf{x}) : \hat{\underline{d}}(\mathbf{x}) \, d\Omega + \int_{\Gamma} [\hat{\mathbf{U}}(\mathbf{x})] \cdot \underline{\sigma}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \, d\Sigma = Q_i(\underline{\sigma}) \cdot \dot{q}_i(\hat{\mathbf{U}}) = \mathbf{Q}(\underline{\sigma}) \cdot \hat{\mathbf{q}}(\hat{\mathbf{U}}) \quad (3.1)$$

*where the applications*

$$\underline{\sigma} \rightarrow \mathbf{Q} = \mathbf{Q}(\underline{\sigma}) \in \mathbb{R}^n \quad (3.2)$$

$$\hat{\mathbf{U}} \rightarrow \hat{\mathbf{q}} = \hat{\mathbf{q}}(\hat{\mathbf{U}}) \in \mathbb{R}^n \quad (3.3)$$

*are linear.*

The  $Q_i$ , components of the  $n$ -dimension vector  $\mathbf{Q}$ , are the *loading* parameters of the problem,  $\mathbf{Q}$  being called the *load*. The  $q_i$ , components of the  $n$ -dimension vector  $\hat{\mathbf{q}}$ , are the associated kinematic parameters.

It is recalled that the statically admissible stress fields  $\underline{\sigma}$  in Eq. (3.1) are piecewise continuous with continuous derivatives, and satisfy the equilibrium equations on  $\Omega$  and the boundary conditions on the stresses on  $\partial\Omega$ . The kinematically admissible virtual velocity field are piecewise continuous with continuous derivative on  $\Omega$  and satisfy the boundary conditions on the velocity on  $\partial\Omega$ ; it must be emphasized that no restriction are imposed on the velocity discontinuities  $[\hat{\mathbf{U}}]$ .

Classical examples of loading parameters are given in [36] ; de Buhan [38], dealing with systems made of composite materials through the homogenization theory, gives interesting developments on the concept of loading parameters regarding both the structure itself and the unit representative cell.

### **3.3.- Statement of the problem.**

The general formulation of the yield design problem is similar to that given in Section (2.1) in the particular case of the truss in Fig.2.

The system under consideration is geometrically defined through  $\Omega$ . It is loaded according to a loading mode depending on  $n$  parameters  $Q_i$ . The strength condition, homologous to Ineq. (2.1) must then be defined. Obviously, such a condition will now refer to the value of the stress tensor  $\underline{\sigma}(\mathbf{x})$  at each point  $\mathbf{x}$  in  $\Omega$ .

The strength domain  $G(x)$  defining the admissible stress states at point  $x$  is given as the only data to characterize the constituent material mechanically. It depends explicitly on  $x$  since the material needs not be homogeneous. The strength condition will be written :

$$\forall x \in \Omega, \quad \sigma(x) \in G(x) \subset \mathbb{R}^6 \quad (3.4)$$

As a rule  $G(x)$  exhibits the following properties

- $\sigma(x) = \underline{0} \in G(x)$  ,
- $G(x)$  is star-shaped with respect to  $\underline{0}$ .

As a matter of fact  $G(x)$  is usually convex (that implies the latter property), which will be assumed from now on.

The question is then to decide whether the system can or cannot withstand a given value of load  $Q$ , under the constraint of the strength condition (3.4).

#### 3.4.- Potentially safe loads, extreme loads.

The answer follows from the same considerations as in chapter 2.

For the system to withstand load  $Q$  under the strength condition (3.4) it is necessary that :

$$\left. \begin{array}{l} \exists \sigma, \text{ statically admissible for the problem} \\ \text{in the loading mode with load } Q, \\ \text{complying with the strength condition (3.4).} \end{array} \right\} \quad (3.5)$$

Under such a load  $Q$ , the system will be said *potentially stable*, and the load itself will be called *potentially safe*.

The set generated by all potentially safe loads will be denoted  $K$  :

$$Q \in K \subset \mathbb{R}^n \Leftrightarrow \exists \sigma \left| \begin{array}{l} \text{S.A. equilibrating } Q \\ \sigma(x) \in G(x), \quad \forall x \in \Omega; \end{array} \right. \quad (3.6)$$

stress field  $\sigma$  is associated with load  $Q$  through the linear application (3.2).

The properties of  $K$  are derived immediately from definition (3.6) and the linearity of application (3.2). It comes out that :

$$\begin{array}{l} Q = \underline{0} \in K, \\ K \text{ is convex,} \end{array} \quad (3.7)$$

what proves that the properties of the  $G$ 's at the level of the constituent material are transferred to  $K$  at the level of the system.

The loads on the convex boundary of  $K$  are called the *extreme loads* of the system, which recalls that any load out of  $K$  is *certainly unsafe* and will induce instability of the system.

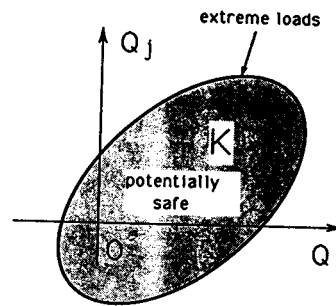


Figure 5 - Domain  $K$  : potentially safe loads, extreme loads.

### 3.5.- Comments.

Comments homologous to those expressed in section (2.3) should be made. The answer illustrated in Fig.5 is but a partial one to the original question : domain  $K$  offers no guarantee of stability for the system, which would require the data of the initial state, of the loading history, and of the mechanical behaviour of the constituent material.

It is hoped that the above given presentation made it clear that the definition of domain  $K$  is only derived from the

$$\text{Compatibility between } \begin{cases} \text{equilibrium (of the system)} \\ \text{resistance (of the material)} \end{cases} \quad (3.8)$$

no other data being available. Therefore, what may appear a shortcoming in the result can also be considered positive : domain  $K$  is independent of the initial state, of the loading history of the system and of the mechanical behaviour of the constituent material apart from the strength condition (3.4). Since this information may be either totally or partially missing in some cases, or improperly assessed in others, it is important that such a general result as  $K$  be available.

Anyhow the question of the relevance of  $K$  for practical applications cannot be disregarded.

From the theoretical point of view it has been proved [39] (cf. [40]) that a system whose constituent material is elastic and perfectly plastic and obeys the maximum plastic work principle will withstand any load  $Q$  inside  $K$  (which generalizes the result in Fig. 4a). In that case the relevance of  $K$  is established.

This theoretical situation is seldom encountered in practice and the relevance of  $K$  must be assessed by experiments and experience. The general idea already expressed in para (2.3.2.), that the deformations needed to fully mobilize the required strengths in the element of the system must be compatible, should be retained. Attention should also be paid to the initial geometrical assumptions and the incidence of geometry changes should be considered.

As an example it appears that in Soil Mechanics, for usual stability analysis purposes, the reasoning based upon the compatibility stated in (3.8) which is the only and very argument of the Yield Design Theory, has proved to be relevant in most cases, but is not valid when dealing for instance with the stability of deep tunnels or deep cavities.

Such validations of the theory support the determination of  $K$  and, more precisely, that of the extreme loads.

### 3.6.- Static approach from inside.

The construction of domain  $K$  comes directly from definition (3.6) : any stress-field  $\underline{\sigma}$ , statically admissible in the loading mode and complying with the strength condition in  $\Omega$  produces a potentially safe load  $Q = Q(\underline{\sigma})$  through application (3.2). In other words :

$$\left. \begin{array}{l} \underline{\sigma} \text{ S.A. in the mode} \\ \underline{\sigma}(\mathbf{x}) \in G(\mathbf{x}) , \forall \mathbf{x} \in \Omega \end{array} \right| \begin{array}{l} \xrightarrow{(3.2)} \\ \rightarrow Q(\underline{\sigma}) \in K \end{array} \quad (3.9)$$

Drawing the convex envelope of such  $Q$  gives an interior approach of  $K$  and "lower bounds" for the extreme loads (Fig. 6).

The exact determination of  $K$  requires exploring the whole set of  $\underline{\sigma}$  satisfying (3.9). This is most often impossible from the practical point of view, except for the case of

simple structures (as in Section 2). It follows that one will usually be satisfied with the interior approach derived through the implementation of construction (3.9) with a reduced number of stress-fields chosen for an efficient balance between simplicity in the procedure and quality of the obtained results.

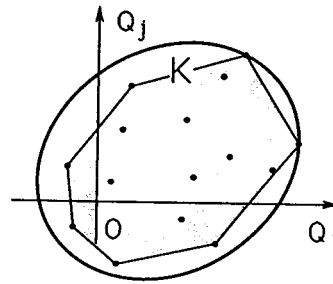


Figure 6 - Static approach from inside.

#### 4.- AN EXAMPLE OF THE YIELD DESIGN STATIC APPROACHES.

##### 4.1.- Stability analysis of a vertical cut.

The stability of the homogeneous vertical cut represented in Fig. 7 is investigated. The load is defined by the gravity forces acting in the bulk of soil, the corresponding loading parameter being the voluminal weight usually denoted  $\gamma$  in Soil Mechanics.

The strength condition of the soil is characterized through Coulomb's isotropic strength criterion with cohesion  $C$  and friction angle  $\phi$ , written in the form

$$f(\underline{\sigma}(\mathbf{x})) = \sup_{i,j=1,2,3} \{ \sigma_i (1 + \sin\phi) - \sigma_j (1 - \sin\phi) - 2 C \cos\phi \} \leq 0 \quad (4.1)$$

where  $\sigma_i$  denote the principal values of  $\underline{\sigma}(\mathbf{x})$ .

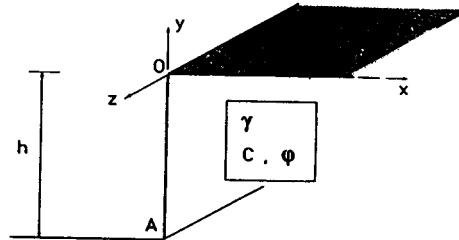


Figure 7 - Stability of a homogeneous vertical cut under its own weight.

#### 4.2.- Static approach from inside.

A stress-field for the static approach from inside was given by Drucker and Prager [41] and proved to comply with Ineq. (4.1) as long as

$$(\gamma h/C) \leq 2 \tan(\pi/4 + \phi/2) ; \quad (4.2)$$

it is presented in Fig.8.

It follows that the value  $2 \tan(\pi/4 + \phi/2)$  is a lower bound for the extreme value of the non-dimensional parameter  $(\gamma h/C)$  which controls the stability of the cut :

$$(\gamma h/C)^+ \geq 2 \tan(\pi/4 + \phi/2) . \quad (4.3)$$

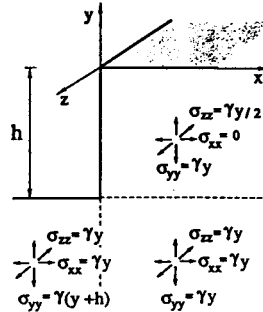


Figure 8 - Static approach for the problem of the vertical cut [41].

#### 4.3.- Static approach from outside.

Still starting from definition (3.6) of K it will now be shown that the static reasoning may be performed in a different way in order to provide an upper bound for  $(\gamma h/C)^+$ .

Let the bulk of soil be virtually separated into two parts (1) and (2) by means of an arbitrary plane (P) passing through the bottom line of the cut as drawn in Fig.9. The compatibility stated in (3.8) implies that, for any such plane (P), the global equilibrium –i.e. from the statics of rigid bodies point of view– of volume (1) or volume (2) be possible under the action of the vertical forces due to gravity (bulk weight of the soil) and of the resisting forces developed by the material.

The cut being considered of infinite length along Oz, the resisting forces developed on the sections parallel to OAB at both ends at infinity are negligible compared with



those along the section by plane (P) : the analysis can therefore be restricted to checking the stability of a slice with thickness  $D$  along  $Oz$  as sketched in Fig.9 assuming no resisting forces on  $OAB$  and  $O'A'B'$ .

Considering the equilibrium of volume (1) submitted to gravity forces acting throughout the volume and to normal and tangential stresses acting on the surface  $ABB'A'$ , the following equations shall be satisfied :

$$\frac{\gamma h^2 D}{2} \tan \alpha \sin \alpha = - \int_{ABB'A'} \sigma dS \quad (4.4)$$

$$\frac{\gamma h^2 D}{2} \sin \alpha t = - \int_{ABB'A'} \tau dS \quad (4.5)$$

where  $\sigma$  and  $\tau$  (a vector) are the normal and tangential components of the stress vector acting at each point of  $ABB'A'$ , tensile stresses are counted positive, and  $t$  is the unit vector along  $BA$ .

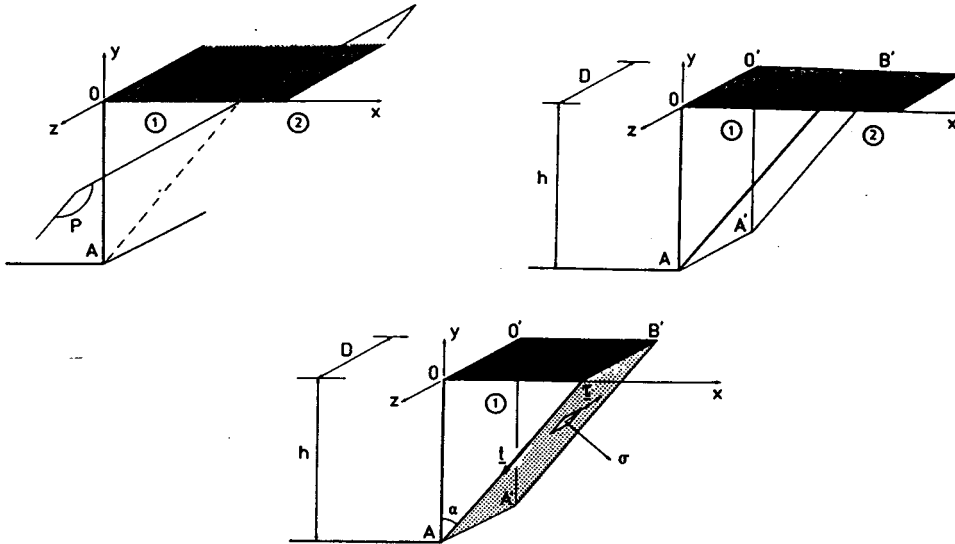


Figure 9 - Global equilibrium analysis of volume (1).

Coulomb's strength condition (4.1) implies that whatever the point in  $ABB'A'$ ,  $\sigma$  and  $\tau$  must comply with the inequality

$$|\tau| \leq C - \sigma \tan \phi \quad (4.6)$$

It follows from Eq. (4.5) and (4.6) that

$$\frac{\gamma h^2 D}{2} \sin \alpha \leq \int_{ABCA} |\tau| dS \leq \int_{ABCA} (C - \sigma \tan \phi) dS \quad (4.7)$$

then through Eq. (4.4)

$$\gamma h/C \leq 2 \cos \phi / \sin \alpha \cos (\alpha + \phi) \quad (4.8)$$

establishing a necessary condition for the stability of the cut, to be satisfied whatever the plane (P).

The minimum of the second hand of Ineq. (4.8) with respect to  $\alpha$  which provides the strongest necessary condition on  $\gamma h/C$ , is obtained for  $\alpha = \pi/4 - \phi/2$  which yields the inequality

$$\gamma h/C \leq 4 \tan(\pi/4 + \phi/2) \quad (4.9)$$

with the conclusion that *any cut for which  $(\gamma h/C)$  is greater than  $4 \tan(\pi/4 + \phi/2)$  will be unstable*; thence :

$$(\gamma h/C)^+ \leq 4 \tan(\pi/4 + \phi/2) \quad (4.10)$$

#### 4.4.- Comments.

It must be emphasized that this latter result is an upper bound for the extreme value  $(\gamma h/C)^+$ . This is often confusing since, being of the "static" type, one would readily expect this approach to be an interior one and to yield a lower bound. It is therefore important to point out where the difference comes from, which may be expressed as follows.

Compatibility between equilibrium (of the system) and resistance (of the material), instead of being thoroughly checked as in (3.6) or (3.9) is only investigated partially, regarding the resulting force in the global equilibrium of an arbitrary triangular shaped volume. Complying with this partial requirements does not automatically ensure that conditions (3.9) will be satisfied and, therefore, does not provide an interior approach of K, i.e. a lower bound for  $(\gamma h/C)^+$ . Conversely, not complying with this requirement implies that conditions (3.9) cannot be satisfied simultaneously, which yields an exterior approach of K, i.e. an upper bound for  $(\gamma h/C)^+$ .

The importance of getting such an exterior approach of K is evident from what has been said about the practical application of the static approach form inside in Section

(3.6). It is also apparent on the above given example that such a static approach from outside

- requires great care in its implementation,
- remains rather crude : It would certainly be worth carrying on thoroughly with the same idea, within the framework of continuum mechanics, that means proving incompatibility between equilibrium and resistance without restricting the analysis to "global" considerations.

The following chapter will present a method which meets this latter goal together with being systematic as regards its implementation.

## 5.- PRINCIPLES OF THE THEORY OF YIELD DESIGN : KINEMATIC APPROACH.

### 5.1.- Fundamental statement of the kinematic approach.

Starting from definition (3.6) of domain  $K$ , Eq. (3.1) expressing the principle of virtual work may be written for any potentially safe load  $Q$ , considering one stress field  $\underline{\sigma}$  referred to in (3.6) and any k.a. virtual velocity field  $\hat{U}$  :

$$\int_{\Omega} \underline{\sigma}(\mathbf{x}) : \hat{\underline{d}}(\mathbf{x}) \, d\Omega + \int_{\Gamma} [\hat{U}(\mathbf{x})] \cdot \underline{\sigma}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \, d\Sigma = Q(\underline{\sigma}) \cdot \dot{q}(\hat{U}) = P_o(Q, \hat{U}) \quad (5.1)$$

The following "π functions" will be introduced :

$$\pi(\mathbf{x}, \hat{\underline{d}}(\mathbf{x})) = \text{Sup} \{ \underline{\sigma}'(\mathbf{x}) : \hat{\underline{d}}(\mathbf{x}) \mid \underline{\sigma}'(\mathbf{x}) \in G(\mathbf{x}) \} \quad (5.2)$$

$$\pi(\mathbf{x}, \mathbf{n}(\mathbf{x}), [\hat{U}(\mathbf{x})]) = \text{Sup} \{ [\hat{U}(\mathbf{x})] \cdot \underline{\sigma}'(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \mid \underline{\sigma}'(\mathbf{x}) \in G(\mathbf{x}) \} \quad (5.3)$$

Since stress field  $\underline{\sigma}$  in Eq. (5.1) complies with strength condition (3.4) in  $\Omega$ , it follows from Eq. (5.1 to 5.3) that :

$$\forall Q \in K, \forall \hat{U} \text{ k.a.}, \quad P_o(Q, \hat{U}) = Q \cdot \dot{q}(\hat{U}) \leq \int_{\Omega} \pi(\mathbf{x}, \hat{\underline{d}}(\mathbf{x})) \, d\Omega + \int_{\Gamma} \pi(\mathbf{x}, \mathbf{n}(\mathbf{x}), [\hat{U}(\mathbf{x})]) \, d\Sigma \quad (5.4)$$

The second hand of Ineq. (5.4) will be denoted  $P_{mr}(\hat{U})$  :

$$P_{mr}(\hat{U}) = \int_{\Omega} \pi(\mathbf{x}, \hat{\underline{d}}(\mathbf{x})) \, d\Omega + \int_{\Gamma} \pi(\mathbf{x}, \mathbf{n}(\mathbf{x}), [\hat{U}(\mathbf{x})]) \, d\Sigma \quad (5.5)$$

and called the *maximum resisting power* in virtual velocity field  $\hat{U}$ .

The obtained result will thus be written in the form :

$$\forall \hat{U} \text{ k.a. } , \quad K \subset \{Q \cdot \dot{q}(\hat{U}) - P_{mr}(\hat{U}) \leq 0\} \quad (5.6)$$

## 5.2.- " $\pi$ functions".

### 5.2.1.- Mechanical significance of the $\pi$ functions.

Statement (5.6) clearly relies on the introduction of the  $\pi$  functions as defined by Eq. (5.2). The significance of these functions must now be investigated.

Eq. (5.2) shows that, given a symmetric tensor  $\hat{d}(x)$  which may be interpreted as a virtual strain rate at point  $x$ ,  $\pi(x, \hat{d}(x))$  represents the maximum value of the work  $\sigma'(x) : \hat{d}(x)$  which may be developed by any stress tensor  $\sigma'(x)$  complying with strength condition (3.4) at point  $x$ . Function  $\pi(x, \hat{d}(x))$  therefore appears as the *maximum resisting power density* in strain rate  $\hat{d}(x)$  under strength condition (3.4) defined by  $G(x)$ .

From the mathematical point of view  $\pi(x, \cdot)$  is the support function of convex  $G(x)$  [42] and it may be proved that the data of  $\pi(x, \cdot)$  is equivalent to that of  $G(x)$  (through a strength criterion for instance), which means that  $\pi(x, \cdot)$  carries, in a dual formulation, all information contained in  $G(x)$ .

The origin of the word "resisting" in the terminology is evident from Eq. (3.1) written as a balance between the power of the external forces and the resisting power (opposite to the power of the internal forces).

The same interpretation as given above holds for  $\pi(x, n(x), [\hat{U}(x)])$  since it may be easily verified that

$$\pi(x, n(x), [\hat{U}(x)]) = \frac{1}{2} \pi(x, n(x) \otimes [\hat{U}(x)] + [\hat{U}(x)] \otimes n(x)) \quad (5.7)$$

It is worth pointing out that the fundamental idea of the kinematic approach, that is defining the strength domain by duality through the  $\pi$  function, is apparent in a paper by Prager [43].

### 5.2.2.- Calculation of the $\pi$ functions.

The values of the  $\pi$  functions are obtained from Eq. (5.2) and (5.3). Considering Eq. (5.7), only the case of function  $\pi(x, \hat{d}(x))$  will be investigated.

Fig.10, schematically presents the two typical circumstances which are encountered when calculating  $\pi(\mathbf{x}, \hat{\mathbf{d}}(\mathbf{x}))$ .

1°- If convex  $G(\mathbf{x})$  is bounded in any direction of  $\mathbb{R}^6$ , then :

$$\pi(\mathbf{x}, \hat{\mathbf{d}}(\mathbf{x})) = \underline{\sigma}^*(\mathbf{x}) : \hat{\mathbf{d}}(\mathbf{x}) \quad (5.8)$$

where  $\underline{\sigma}^*(\mathbf{x})$  is any stress tensor on the boundary of  $G(\mathbf{x})$  where the outward normal is colinear to  $\hat{\mathbf{d}}(\mathbf{x})$ . It follows that, in that case,  $\pi(\mathbf{x}, \hat{\mathbf{d}}(\mathbf{x}))$  is finite whatever  $\hat{\mathbf{d}}(\mathbf{x})$ .

2°- If convex  $G(\mathbf{x})$  is not bounded in any direction, let  $(I)$  be the convex cone of the directions in which it extends to infinity in  $\mathbb{R}^6$ . Then :

• If  $\hat{\mathbf{d}}(\mathbf{x})$  belongs to the convex cone orthogonal to  $(I)$ ,  $\pi(\mathbf{x}, \hat{\mathbf{d}}(\mathbf{x}))$  has a finite value

$$\pi(\mathbf{x}, \hat{\mathbf{d}}(\mathbf{x})) = \underline{\sigma}^*(\mathbf{x}) : \hat{\mathbf{d}}(\mathbf{x}) \quad (5.9)$$

where  $\underline{\sigma}^*(\mathbf{x})$  is any stress tensor on the boundary of  $G(\mathbf{x})$  where the outward normal is colinear to  $\hat{\mathbf{d}}(\mathbf{x})$ , as in the preceding case ;

• if  $\hat{\mathbf{d}}(\mathbf{x})$  does not belong to that cone,  $\pi(\mathbf{x}, \hat{\mathbf{d}}(\mathbf{x}))$  is infinite

$$\pi(\mathbf{x}, \hat{\mathbf{d}}(\mathbf{x})) = +\infty \quad (5.10)$$



Figure 10 - Calculation of  $\pi(\mathbf{x}, \hat{\mathbf{d}}(\mathbf{x}))$ .

Within the framework of classical continuum mechanics, strength domains under general three dimensional stress condition are usually of the 2nd type presented in Fig. 10 (e.g. Tresca's, von Mises', Coulomb's, ... strength criteria), and the corresponding  $\pi$  functions take finite or infinite values according to the given  $\hat{\mathbf{d}}(\mathbf{x})$ . It is obvious that only in the case when  $G(\mathbf{x})$  is unbounded in any direction, that is the ideal rigid material, will  $\pi(\mathbf{x}, \hat{\mathbf{d}}(\mathbf{x}))$  be infinite whatever  $\hat{\mathbf{d}}(\mathbf{x})$ .

Strength domains for plane stress loading of the material are bounded in any direction, as are the strength domains referring to bending moments in the yield design

of plates and the strength domains for bending moments, normal and shear forces, etc. in the yield design of beams.

A comprehensive, but of course non exhaustive, list of strength criteria and corresponding  $\pi$  functions is given in [36]. A shorter list appears in Section 5.4.

### 5.2.3.- Properties of the $\pi$ functions.

The following properties are evident from Eq. (5.2) and the assumptions on  $G(x)$  :

- $\pi(x, \cdot)$  is non negative,
- $\pi(x, \cdot)$  is positively homogeneous of degree 1,
- $\pi(x, \cdot)$  is convex.

### 5.3.- The kinematic approach from outside.

#### 5.3.1.- Exterior approach.

Statement (5.6) provides a powerful exterior approach of K.

For any virtual k.a. velocity field  $\hat{U}$  the maximum resisting power  $P_{mr}(\hat{U})$  is computed referring to the relevant expressions of the  $\pi$  functions, while the power of the external forces  $P_e(Q, \hat{U}) = Q \cdot \dot{q}(\hat{U})$  is a linear form in  $Q$  where  $\dot{q}(U)$  is known. It follows that domain K is included in the halfspace of  $R^n$  defined by Ineq. (5.6).

Repeating the procedure with different  $\dot{q}$  rapidly gives an *exterior approach* of the boundary of K and *upper bounds* for the extreme loads (Fig. 11).

Furthermore it was proved, with some complementary mathematical assumptions, that the exact dual definition of K is given by Form. (5.6), meaning that K can be generated by applying Form. (5.6) to all k.a. virtual velocity fields [44-46].

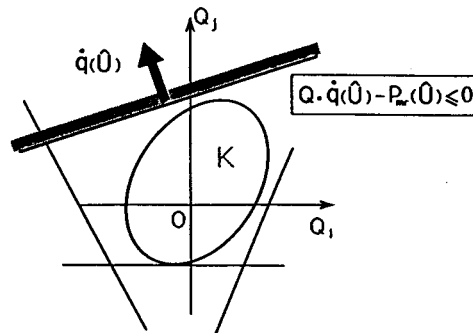


Figure 11 - Kinematical approach from outside.

### 5.3.2.- Relevant velocity fields.

For Ineq. (5.6) to give a non trivial result the k.a. virtual velocity field must be chosen in such a way that :

- the position of the external forces be non zero
- the maximum resisting power be finite.

The first condition does not need any comment since it meets common sense. The latter means that  $\hat{U}$  should be selected in order to obtain finite values for the corresponding  $\pi$  functions (cf. para. 5.2.2.).

Such virtual velocity fields will be said *relevant*. It must be emphasized that this concept comes straight from the dual formulation of the strength condition and has nothing to do with any flow rule or other constitutive assumption regarding the way the material deforms or collapses when the strength condition is saturated.

### 5.4.- Some Classical strength criteria and corresponding $\pi$ functions.

#### 5.4.1.- Definition of $G(x)$ through a strength criterion.

As a rule, the strength domain  $G(x)$  is characterized, from the practical point of view, by the data of a strength criterion through a scalar function  $f$  of  $\underline{\sigma}(x)$ , which depends explicitly on  $x$  as does  $G(x)$ , such that :

$$\begin{aligned} f(x, \underline{\sigma}(x)) < 0 & \Leftrightarrow \underline{\sigma}(x) \text{ interior to } G(x) , \\ f(x, \underline{\sigma}(x)) = 0 & \Leftrightarrow \underline{\sigma}(x) \text{ on the boundary of } G(x) , \\ f(x, \underline{\sigma}(x)) > 0 & \Leftrightarrow \underline{\sigma}(x) \text{ exterior to } G(x) , \end{aligned} \quad (5.11)$$

For practical applications, function  $f(x, \cdot)$  is usually chosen a convex, continuous, with piecewise continuous derivatives, function of  $\underline{\sigma}(x)$ . It is clear anyhow that such a function  $f(x, \cdot)$  for a given  $G(x)$  is not unique.

With this definition of  $G(x)$ , Eq. (5.2) may now be written, with simplified notations, in the form :

$$\pi(x, \underline{d}) = \sup_{\underline{\sigma}} \{ \underline{\sigma} : \underline{d} \mid f(x, \underline{\sigma}) \leq 0 \} . \quad (5.12)$$

Conversely, the data of function  $\pi(x, \cdot)$  as the dual definition of  $G(x)$  provides an expression for function  $f(x, \cdot)$  :

$$f(x, \underline{\sigma}) = \sup_{\underline{d}} \{ \underline{\sigma} : \underline{d} - \pi(\underline{d}) \mid \text{tr } \underline{d}^2 = 1 \} \quad (5.13)$$

this expression may not be the most popular one for the considered  $G(x)$  as regards practical applications.

Staying within the framework of classical continuum mechanics some usual strength criteria and corresponding  $\pi$  functions for isotropic materials will now be presented, together with the strength criteria and  $\pi$  functions of the most commonly encountered isotropic interfaces. In order to simplify the expressions, the dependence on  $\mathbf{x}$  will not be explicit and the notation  $\mathbf{V}$  will be used for  $[\hat{\mathbf{U}}]$ .

**5.4.2.- Von Mises's strength criterion and  $\pi$  functions.**

$$f(\underline{\sigma}) = \sqrt{\text{tr } \underline{s}^2 / 2} - k \quad (5.14)$$

where  $\underline{s}$  is the deviatoric part of tensor  $\underline{\sigma}$ .

$$\begin{aligned} \pi(\underline{d}) &= +\infty & \text{if } \text{tr } \underline{d} \neq 0, \\ \pi(\underline{d}) &= k \sqrt{2 \text{tr } \underline{d}^2} & \text{if } \text{tr } \underline{d} = 0; \end{aligned} \quad (5.15a)$$

thence :

$$\begin{aligned} \pi(\mathbf{n}, \mathbf{V}) &= +\infty & \text{if } \mathbf{V} \cdot \mathbf{n} \neq 0, \\ \pi(\mathbf{n}, \mathbf{V}) &= k |\mathbf{V}| & \text{if } \mathbf{V} \cdot \mathbf{n} = 0. \end{aligned} \quad (5.15b)$$

**5.4.3.- Tresca's strength criterion and  $\pi$  functions.**

$$f(\underline{\sigma}) = \sup_{i,j=1,2,3} \{\sigma_i - \sigma_j - \sigma_0\} \quad (5.16)$$

where  $\sigma_i, \sigma_j$  denote the principal stresses.

$$\begin{aligned} \pi(\underline{d}) &= +\infty & \text{if } \text{tr } \underline{d} \neq 0, \\ \pi(\underline{d}) &= \frac{\sigma_0}{2} (|d_1| + |d_2| + |d_3|) & \text{if } \text{tr } \underline{d} = 0; \end{aligned} \quad (5.17a)$$

where the  $d_i$ 's are the principal components of  $\underline{d}$ ; thence :

$$\begin{aligned} \pi(\mathbf{n}, \mathbf{V}) &= +\infty & \text{if } \mathbf{V} \cdot \mathbf{n} \neq 0, \\ \pi(\mathbf{n}, \mathbf{V}) &= \frac{\sigma_0}{2} |\mathbf{V}| & \text{if } \mathbf{V} \cdot \mathbf{n} = 0. \end{aligned} \quad (5.17b)$$

**5.4.4.- Coulomb's strength criterion and  $\pi$  functions.**

$$f(\underline{\sigma}) = \sup_{i,j=1,2,3} \{\sigma_i(1 + \sin\varphi) - \sigma_j(1 - \sin\varphi) - 2 C \cos\varphi\} \quad (5.18)$$

$$\begin{aligned} \pi(\underline{d}) &= +\infty & \text{if } \text{tr } \underline{d} < (|d_1| + |d_2| + |d_3|) \sin\varphi, \\ \pi(\underline{d}) &= C \cot\varphi \text{tr } \underline{d} & \text{if } \text{tr } \underline{d} \geq (|d_1| + |d_2| + |d_3|) \sin\varphi; \end{aligned} \quad (5.19a)$$

thence :

$$\begin{aligned} \pi(\mathbf{n}, \mathbf{V}) &= +\infty & \text{if } \mathbf{V} \cdot \mathbf{n} < |\mathbf{V}| \sin\varphi, \\ \pi(\mathbf{n}, \mathbf{V}) &= C \cot\varphi \mathbf{V} \cdot \mathbf{n} & \text{if } \mathbf{V} \cdot \mathbf{n} \geq |\mathbf{V}| \sin\varphi. \end{aligned} \quad (5.19b)$$



It will be observed that these expressions are different from the homologous formulae given by Chen [19] which are only valid in the case when the inequalities in (5.19) are saturated.

5.4.5. - *Drucker-Prager's criterion and  $\pi$  functions.*

$$f(\underline{\sigma}) = \sqrt{\text{tr } \underline{\sigma}^2 / 2} - \frac{3 \sin \varphi}{\sqrt{3(3 + \sin^2 \varphi)}} (C \cot \varphi - \text{tr } \underline{\sigma} / 3) \quad (5.20)$$

$$\begin{aligned} \pi(\underline{d}) &= +\infty & \text{if } \text{tr } \underline{d} < \sqrt{\frac{2 \sin^2 \varphi}{3 + \sin^2 \varphi} (3 \text{tr } \underline{d}^2 - (\text{tr } \underline{d})^2)} \\ \pi(\underline{d}) &= C \cot \varphi \text{tr } \underline{d} & \text{if } \text{tr } \underline{d} \geq \sqrt{\frac{2 \sin^2 \varphi}{3 + \sin^2 \varphi} (3 \text{tr } \underline{d}^2 - (\text{tr } \underline{d})^2)} \end{aligned} \quad (5.21a)$$

thence :

$$\begin{aligned} \pi(\underline{n}, \underline{V}) &= +\infty & \text{if } \underline{V} \cdot \underline{n} < |\underline{V}| \sin \varphi \\ \pi(\underline{n}, \underline{V}) &= C \cot \varphi \underline{V} \cdot \underline{n} & \text{if } \underline{V} \cdot \underline{n} \geq |\underline{V}| \sin \varphi \end{aligned} \quad (5.21b)$$

5.4.6.- *Tresca's strength criterion with T-tension cut off and  $\pi$  functions.*

$$f(\underline{\sigma}) = \sup_{i,j=1,2,3} \{ \sigma_i - \sigma_j - \sigma_0, \sigma_i - T \} \quad (5.22)$$

$$\begin{aligned} \pi(\underline{d}) &= +\infty \\ \pi(\underline{d}) &= \frac{\sigma_0}{2} (|d_1| + |d_2| + |d_3|) + (T - \frac{\sigma_0}{2}) \text{tr } \underline{d} & \text{if } \text{tr } \underline{d} < 0 \\ & & \text{if } \text{tr } \underline{d} \geq 0 \end{aligned} \quad (5.23a)$$

thence :

$$\begin{aligned} \pi(\underline{n}, \underline{V}) &= +\infty & \text{if } \underline{V} \cdot \underline{n} < 0 \\ \pi(\underline{n}, \underline{V}) &= \frac{\sigma_0}{2} |\underline{V}| + (T - \frac{\sigma_0}{2}) \underline{V} \cdot \underline{n} & \text{if } \underline{V} \cdot \underline{n} \geq 0 \end{aligned} \quad (5.23b)$$

When the material under consideration does not withstand any tensile stress the value of T is set equal to nought.

5.4.7.- *Coulomb's strength criterion with T-tension cut off and  $\pi$  functions.*

$$f(\underline{\sigma}) = \sup_{i,j=1,2,3} \{ \sigma_i (1 + \sin \varphi) - \sigma_j (1 - \sin \varphi) - 2 C \cos \varphi, \sigma_i - T \} \quad (5.24)$$

$$\begin{aligned}
\pi(\underline{d}) &= +\infty && \text{if } \text{tr } \underline{d} < (|d_1| + |d_2| + |d_3|) \sin\varphi, \\
\pi(\underline{d}) &= \frac{C \cos\varphi - T \sin\varphi}{1 - \sin\varphi} (|d_1| + |d_2| + |d_3|) + \frac{T - C \cos\varphi}{1 - \sin\varphi} \text{tr } \underline{d} && (5.25a) \\
&&& \text{if } \text{tr } \underline{d} \geq (|d_1| + |d_2| + |d_3|) \sin\varphi;
\end{aligned}$$

thence :

$$\begin{aligned}
\pi(\mathbf{n}, \mathbf{V}) &= +\infty && \text{if } \mathbf{V} \cdot \mathbf{n} < |\mathbf{V}| \sin\varphi \\
\pi(\mathbf{n}, \mathbf{V}) &= \frac{C \cos\varphi - T \sin\varphi}{1 - \sin\varphi} |\mathbf{V}| + \frac{T - C \cos\varphi}{1 - \sin\varphi} \mathbf{V} \cdot \mathbf{n} && (5.25b) \\
&&& \text{if } \mathbf{V} \cdot \mathbf{n} \geq |\mathbf{V}| \sin\varphi;
\end{aligned}$$

#### 5.4.8.- Smooth or frictionless interface.

Vector  $\mathbf{T}$  denotes the stress vector, with components  $(\sigma, \tau)$ , acting on the interface with outward normal  $\mathbf{n}$ . Vector  $\mathbf{V}$  stands for the velocity jump in the interface.

$$f(\mathbf{T}) = \text{Sup } (\sigma, \tau) \quad (5.26)$$

$$\begin{aligned}
\pi(\mathbf{V}) &= +\infty && \text{if } \mathbf{V} \cdot \mathbf{n} < 0, \\
\pi(\mathbf{V}) &= 0 && \text{if } \mathbf{V} \cdot \mathbf{n} \geq 0.
\end{aligned} \quad (5.27)$$

#### 5.4.9.- Isotropic interface with Tresca's friction and no resistance to traction.

$$f(\mathbf{T}) = \text{Sup } \{\sigma, \tau - C_i\} \quad (5.28)$$

where  $C_i$  denotes the shear strength of the interface.

$$\begin{aligned}
\pi(\mathbf{V}) &= +\infty && \text{if } \mathbf{V} \cdot \mathbf{n} < 0, \\
\pi(\mathbf{V}) &= C_i V_t && \text{if } \mathbf{V} \cdot \mathbf{n} \geq 0
\end{aligned} \quad (5.29)$$

where  $V_t$  is the modulus of the component of  $\mathbf{V}$  in the interface.

#### 5.4.10.- Isotropic interface with Coulomb's dry friction.

$$f(\mathbf{T}) = \tau - \sigma \tan\varphi_i \quad (5.30)$$

where  $\varphi_i$  denotes the friction angle of the interface.

$$\begin{aligned}
\pi(\mathbf{V}) &= +\infty && \text{if } V_n < V_t \tan\varphi_i, \\
\pi(\mathbf{V}) &= 0 && \text{if } V_n \geq V_t \tan\varphi_i.
\end{aligned} \quad (5.31)$$

### 5.5.- Implementation of the kinematic approach from outside.

#### 5.5.1.- Classical applications.

Usually the practical implementation of the kinematic approach from outside makes use of *simple* relevant velocity fields depending on a reduced number of parameters. The optimization of the obtained results is gained through the minimization of  $P_{mr}(\hat{\mathbf{U}})$  with respect to these parameters while keeping  $\dot{\mathbf{q}}(\hat{\mathbf{U}})$  constant. Examples of such classically used velocity fields may be given now.

Piecewise rigid body motion virtual velocity fields where one rigid block is moving and the rest of the structure is motionless, or more elaborated virtual "mechanisms" where several rigid blocks move with respect to one another, will be encountered (Fig. 12). In both cases the virtual velocity jumps will be governed by the condition that  $\pi(\mathbf{x}, \mathbf{n}(\mathbf{x}), [\hat{\mathbf{U}}(\mathbf{x})])$  remains finite as explained in para. (5.3.2)

- Tresca's or von Mises' strength criterion -

For instance, when dealing with a material whose strength criterion is of the Tresca or von Mises types, it comes out from Eq. (5.15) and (5.17) that such k.a. relevant virtual velocity fields will only exhibit purely tangential velocity jumps ; examples are shown in Fig. 12:

$$[\hat{\mathbf{U}}(\mathbf{x})] \cdot \mathbf{n}(\mathbf{x}) = 0 . \quad (5.32)$$

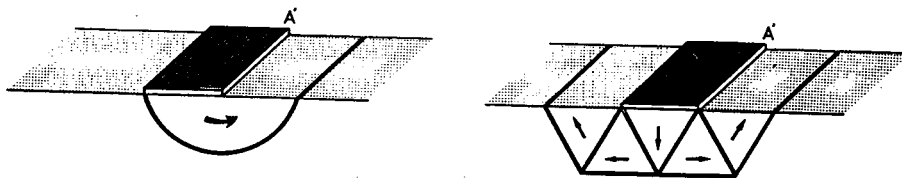


Figure 12 - Indentation of a half space by a rigid plate : relevant rigid body motion virtual velocity fields in the case of Tresca's or von Mises' strength criterion.

Relevant virtual velocity fields will also be constructed where the strain rate field is non zero : they will be governed by the condition that  $\pi(\mathbf{x}, \hat{\mathbf{d}}(\mathbf{x}))$  be finite.

Still taking the case of a Tresca or von Mises strength criterion as an example, the corresponding k.a. relevant virtual velocity fields will induce no volume change as can be seen from Eq. (5.15) and (5.17) :

$$\text{tr } \hat{\mathbf{d}}(\mathbf{x}) = 0 . \quad (5.33)$$

Such fields as shear velocity fields will thus be relevant for those materials.

- Coulomb's strength condition -

The case of a material with a strength criterion of Coulomb's type, must now be considered for it will help emphasizing some essential points of the kinematic approach. Eq. (5.19) gives the conditions to be satisfied by relevant virtual velocity fields :

$$\text{tr } \hat{\underline{d}}(\underline{x}) \geq (|\hat{d}_1(\underline{x})| + |\hat{d}_2(\underline{x})| + |\hat{d}_3(\underline{x})|) \sin \varphi, \quad (5.34)$$

which establishes a minimum value for the virtual dilatancy ;

$$[\hat{\underline{U}}(\underline{x})] \cdot \underline{n}(\underline{x}) \geq |[\hat{\underline{U}}(\underline{x})]| \sin \varphi, \quad (5.35)$$

which shows that the velocity jump, if non zero, cannot be purely tangential :  $[\hat{\underline{U}}(\underline{x})]$  must be inclined over the jump surface  $\Sigma$  by an angle  $\beta$  at least equal to  $\varphi$  and be directed outwards, corresponding to a virtual separation between both sides of the jump surface (Fig.13).

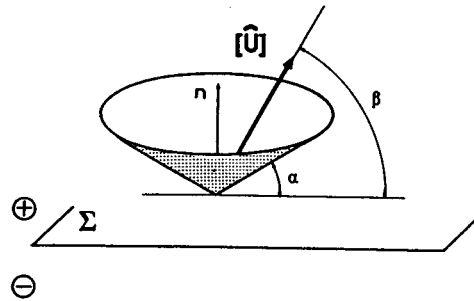


Figure 13 - The velocity discontinuity  $[\hat{\underline{U}}]$  of a relevant k.a. virtual velocity field in the case of Coulomb's strength criterion.

It must be underlined once again that conditions (5.34, 5.35) are imposed on virtual velocity fields and do not claim any significance from the point of view of any flow rule,... ; this is often misunderstood. Moreover it must be recalled that any velocity jump is allowable for a virtual velocity field : velocity jump surfaces need not be slip surfaces where the velocity jump is tangential. Therefore conditions (5.35) and (5.32) are exactly on the same status, proceeding only from the strength condition of the considered material. The latter point has often proved a difficulty in the understanding of the kinematic approach from outside, in soil mechanics for instance, through it is the key of the efficiency of the method.

Obviously from Eq. (5.21, 5.23, 5.25), what has just been said for Coulomb's strength criterion will apply to the strength criteria considered in para. (5.4.5. - 5.4.7.).

- Plane strain relevant k.a. virtual velocity fields -

When the problem under consideration accepts plane strain velocity fields as kinematically admissible the simplification due to passing from dimension 3 to dimension 2 makes it possible to construct more elaborate relevant k.a. virtual velocity fields.

Considering first the case of Tresca's or von Mises' strength criterion (or any criterion independent of  $\text{tr}\alpha$ ), Eq. (5.33) applied to virtual k.a. velocity fields in plane strain parallel to Oxy results in generating the relevant virtual velocity fields in the following way.

Considering two families of mutually orthogonal curves ( $\alpha$ - and  $\beta$ -lines), the components  $\hat{U}_\alpha$  and  $\hat{U}_\beta$  of  $\hat{U}$  at the current point verify the differential equations [47]:

$$\begin{aligned} d\hat{U}_\alpha - \hat{U}_\beta d\theta &= 0 & \text{along the } \alpha\text{-lines} \\ d\hat{U}_\beta + \hat{U}_\alpha d\theta &= 0 & \text{along the } \beta\text{-lines} \end{aligned} \quad (5.36)$$

where  $\theta$  is defined as  $\theta = (0x, e_\alpha)$ ,  $e_\alpha$  and  $e_\beta$  being the unit vectors tangent to the  $\alpha$ - and  $\beta$ -lines at the current point,  $(e_\alpha, e_\beta) = +\pi/2$  (Fig. 14).

For Coulomb's strength criterion, equations homologous to (5.36) are established for the plane strain velocity fields which saturate Ineq. (5.34) (cf. [48]).

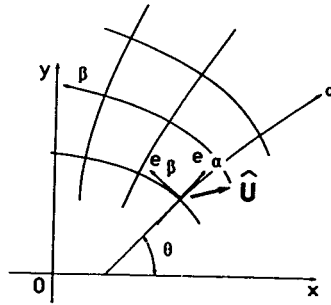
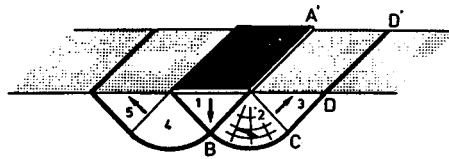


Figure 14 - Plane strain relevant virtual k.a. velocity fields for Tresca's strength criterion : Geiringer's equations [47].

A classical example of such a plane strain relevant k.a. virtual velocity field is recalled in Fig. 15 where volumes (1), (3), and (5) are given rigid body motions while in volumes (2) and (4) the velocity is constant and orthoradial ; the plane and cylindrical surfaces  $ABB'A'$ ,  $BCC'B'$ , and  $CDD'C'$  and the symmetric ones are velocity discontinuity surfaces.



*Figure 15 - Indentation of a half space by a rigid plate : Prandtl's velocity field in the case of Tresca's strength criterion.*

It may be added that, due to the corresponding simplifications, many yield design problems are often treated within the plane strain assumption, at least for a first study. A detailed presentation as to the significance of this treatment within the framework of yield design may be found in [36] and many interesting solutions are available in numerous textbooks.

#### *5.5.2.- Numerical implementation.*

The efficiency of the methods which have just been sketched is highly improved when analytical methods for the construction of k.a. relevant virtual velocity fields are combined with numerical minimization procedures. This has been the basis of computer codes which have been developed for instance for application to soil mechanics stability analysis problems (e.g. [49 - 51]).

Many attempts have also been made during the two past decades aiming at using numerical methods directly for the application of kinematic approach from outside. Finite element methods have been considered for a straight forward application of the theory [52 - 56], which has proved to present difficulties as regards the generation of strictly relevant virtual velocity fields, and the minimization procedure. In many cases the results so obtained have scarcely been as good as those gained through classical procedures where discontinuous relevant virtual velocity fields can usually be generated and explored more easily.

Recently, in the case of Tresca's strength criterion, a numerical method based upon the generation of stream functions has been developed and seems quite efficient [57, 58].

#### 6.- AN EXAMPLE OF THE YIELD DESIGN KINEMATIC APPROACH [59].

##### 6.1.- *Stability of a vertical cut for a purely cohesive soil.*

The stability of the vertical cut introduced in Fig.7 is now considered, assuming the homogeneous constituent soil to be purely cohesive, i.e. with a Trésca strength criterion.

The relevant k.a. virtual velocity fields for such a material have already been discussed in para. (5.5.1).

Applying the same arguments as in Section (4.3) it appears that the kinematic approach for the problem can be restricted to a slice of arbitrary thickness  $D$  parallel to plane  $Oxy$  with no resisting forces nor resisting power developed in the parallel sections at both extremities. (A detailed presentation of this type of yield design problems –plane strain yield design problems– appear in [36] where it is shown how they can be studied through a two-dimensional yield design analysis).

For the slice of thickness  $D$  the following k.a. virtual velocity field can be retained in accordance with Eq. (5.32, 5.33) : the prismatic volume  $OABB'A'O'$  is given a virtual downwards translation motion with velocity  $V$  parallel to  $AB$ , while the rest of the soil mass remains motionless (Fig.16).

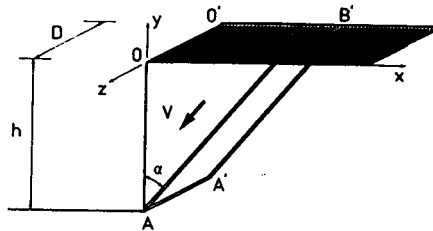


Figure 16 - *Stability of a vertical cut in a purely cohesive soil : kinematic approach.*

The power of the external forces performed in this virtual velocity field by the gravity forces writes :

$$Q\dot{q}(\hat{U}) = \frac{\gamma h^2 D}{2} V \sin \alpha \quad (6.1)$$

while the maximum resisting power being developed along plane ABB'A' in the tangential velocity discontinuity reduces to :

$$P_{mr}(\hat{U}) = CV h D / \cos \alpha \quad (6.2)$$

Thence through Ineq. (5.4)

$$(\gamma h / C)^+ \leq 4 / \sin 2\alpha \quad (6.3)$$

Looking for the minimum of the second hand of this upper bound with respect to  $\alpha$  gives  $\alpha = \pi/4$  and

$$(\gamma h / C)^+ \leq 4 \quad (6.4)$$

as the best upper estimate of  $(\gamma h / C)^+$  for this class of one parameter k.a. virtual velocity fields.

The following comments can be made.

(a) Comparing the upper bound (6.4) with the value obtained in Section (4.3) (Ineq. 4.10) for this particular case where  $\varphi = 0$  shows that both results are identical :

$$(\gamma h / C) \leq 4$$

is proved a necessary condition for the stability of the cut.

(b) This is not a mere coincidence!

As a matter of fact the kinematic reasoning performed here with a uniform translation motion given to volume OABB'A'O' results in checking the global equilibrium of that volume, as regards the resultant component in the direction of the motion, of the gravity forces in  $\Omega$  and resisting forces developed on ABB'A'.

Through the use of function  $\pi(n, [\hat{U}])$  here, the dual procedure in the kinematic approach

- automatically finds out the axis along which the global equilibrium equation will produce a non trivial necessary stability condition due to the expression of the strength condition (here this relevant axis is AB),

- selects the distribution(s) of the resisting forces which most favour(s) the global equilibrium of the volume so that external forces being not balanced under these conditions will necessarily mean the instability of the cut.

(c) One may question about the admissibility of the considered velocity field since the tip AA' of volume OABB'A'O' being given the velocity  $V$  parallel to AB would penetrate the horizontal surface of the motionless mass of the rest of the soil.



Many arguments have been put forward in order to solve this apparent paradox which occurs quite frequently in the applications of the kinematic approach (cf. Fig.12 and 15). For those which are relevant they appear adequate to the problem under consideration with no versatility to be applied to similar cases, and do actually miss the fundamental point.

The answer lies in the very significance of virtual k.a. velocity fields, already recalled in para. (5.5.1), that are the *test functions*, the "mathematical tools" in the dualization procedure. This means that they are piecewise continuous with piecewise continuous derivatives, but no condition is imposed on the possible velocity discontinuity. In the application of the kinematic approach, the "strength" of the material, through the corresponding  $\pi(n, [\hat{U}])$  function, imposed the choice of tangential velocity discontinuity on the plane AA'B'B for the value of  $\pi(n, [\hat{U}])$  to remain finite. *No such condition is to be found to impose any constraint on  $\hat{U}$  along AA'* and this can be understood from the following : the three-dimensional continuum model does not take lineal densities of forces into account (nor does the two-dimensional model with punctual ones) and therefore the static analysis performed in Section (4.3) did not consider any lineal resisting force at the vertex of volume (1) along AA', which is consistent with no term appearing on AA' in the dual formulation.

(d) Another result of the kinematic approach is to answer the question of whether the consideration of the moment equation of the global equilibrium of volume (1) in the static analysis performed at Section (4.3) would have led to an additional limitation on  $(\gamma h/C)$  for the stability necessary condition.

The answer is negative and is apparent from the dual formulation. Considering the moment equation corresponds, in the dual formulation, to a rigid body rotational motion of volume OABB'A'O' inducing a velocity discontinuity across plane ABB'A' which would not be tangential everywhere, unless the axis of rotation parallel to Oz be rejected at infinity in the direction normal to AB which is the case already examined.

This illustrates the fact that, but for that particular case which leads to the final result already obtained, the strength capacities of the soil will have no limiting consequence on the balance of the moment equation of global equilibrium of volume (1).

Thus Ineq. (6.4) expresses the best result to be derived from the consideration of the global equilibrium of triangular prismatic volumes.

(e) The same reasoning, through the kinematic approach as the dual formulation of the static analysis, shows that circular cylindrical volumes may also be considered for the "global equilibrium check" and produce significant results. In that case the relevant equation which provides a necessary stability condition on  $(\gamma h/C)$  turns out to be the moment equation with respect to the geometrical axis of the cylinder (Fig. 17).

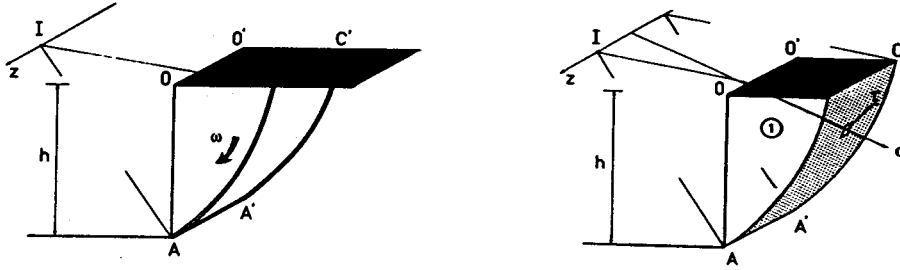


Figure 17 - Stability of a vertical cut in purely cohesive soil : kinematic approach with a rigid body rotational motion and "global equilibrium check".

The result obtained by looking for the strongest necessary condition over all cylindrical volumes is classical [16, 17] and is given by

$$(\gamma h/C)^+ \leq 3.83 \quad (6.5)$$

This upper estimate is better than condition (6.4) ; this is no surprise since the analyses with triangular prismatic volumes are but particular cases of the present ones.

(f) Finally it also appears from the dual formulation that no other cylindrical volumes can be considered significantly for a static analysis through the "global equilibrium check" : they would not result in any further constraint on  $(\gamma h/C)^+$ .

The upper bound resulting from Ineq. (6.5) for  $(\gamma h/C)^+$  has long been the best available for this problem, despite numerous attempts where even sophisticated numerical methods have been used. De Buhan, Dormieux and Maghous [60] have quite recently produced a better result, obtained rigorously without heavy numerical procedure, through an ingenious relevant virtual velocity field :

$$(\gamma h/C)^+ \leq 3.817 \quad (6.6)$$

It is clear that the importance of this result will not be assessed from its consequences as regards practical applications : the analysis of the stability of a vertical cut is somewhat of a symbol where the "slip circle" has been baffling the endeavours of researchers for decades, although it could be proved that it was not to give the exact answer to the problem.

### 6.2.- Stability of a vertical cut for a soil exhibiting both cohesion and friction.

Assuming now the strength condition of the constituent soil to be of Coulomb's type, the following relevant k.a. virtual velocity fields will be used which satisfy Ineq. (5.34, 5.35) : the prismatic volume  $OABB'A'O'$  is given a virtual downward translation motion with the velocity  $V$  inclined at an angle  $\beta$  to  $AB$  with  $\varphi \leq \beta \leq \pi - \varphi$  while the rest of the soil mass remains motionless (Fig. 18).

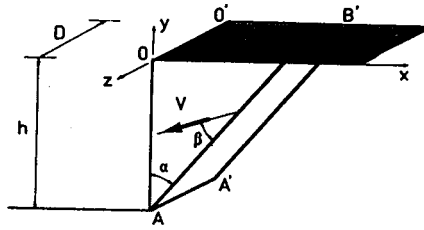


Figure 18 - Stability of a vertical cut in a frictional and cohesive soils : kinematic approach.

The power performed in this velocity field by the gravity forces becomes :

$$Q\dot{q}(\hat{U}) = \frac{\gamma h^2 D}{2} \tan \alpha V \cos(\alpha + \beta) \quad (6.7)$$

while the maximum resisting power being developed along plane  $ABB'A'$  in the velocity discontinuity is equal to

$$P_{mr}(\hat{U}) = \frac{C}{\tan \varphi} \frac{hD}{\cos \alpha} V \sin \beta \quad (6.8)$$

It follows through Ineq. (5.4)

$$\left( \frac{\gamma h}{C} \right)^+ \leq \frac{2 \sin \beta}{\tan \varphi \sin \alpha \cos(\alpha + \beta)} \quad (6.9)$$

where  $\alpha$  and  $\beta$  are two parameters with the constraints

$$\begin{aligned} 0 &\leq \alpha + \beta \leq \pi/2 \\ \varphi &\leq \beta \leq \pi - \varphi \end{aligned} \quad (6.10)$$

Minimizing the second hand of Ineq. (6.9) with respect to  $\beta$  corresponds to

$$\beta = \varphi \quad (6.11)$$

for which Ineq. (6.9) becomes

$$(\gamma h/C)^+ \leq 2 \cos \varphi / \sin \alpha \cos(\alpha + \varphi) \quad (6.12)$$

(to be compared with Ineq. (4.8)).

Minimizing then with respect to  $\alpha$  (as already done at Section 4.3) gives

$$(\gamma h/C)^+ \leq 4 \tan(\pi/4 + \varphi/2) \quad (6.13)$$

as the best estimate of  $(\gamma h/C)^+$  one may achieve for this class of two parameters k.a. virtual velocity fields.

This results also deserves some comments to complete those already made in the simpler case of the purely cohesive soil.

(a) As in the case of a purely cohesive soil it is found that the upper estimate derived from the kinematic approach with the virtual k.a. velocity fields of Fig. 18 is equal to the result derived from the static analysis of Section (4.3) for the same volume (1) in Fig. 9 dealing with the global equilibrium of volume (1) from the point of view of the resulting force.

(b) One may also notice that the optimality condition  $\beta = \varphi$  in the kinematic approach means that, in the static analysis, the distributions of  $(\sigma, \tau)$  along plane ABB'A' which most favour the global equilibrium of volume (1) correspond to the equality

$$|\tau| = C - \sigma \tan \varphi \quad \text{with } \tau \text{ in direction AB}$$

(which is quite evident here).

It also shows that the significant global equilibrium equation which is to provide the stability constraint on  $(\gamma h/C)$  is the resulting force equation in the direction at angle  $\varphi$  to AB (the direction of  $\mathbf{V}$ ).

(c) The optimality condition  $\beta = \varphi$  found in this particular case deserves being looked at in a more general perspective since the constraints on  $[\mathbf{U}]$  for the relevant virtual k.a. velocity fields derived from Ineq. (5.35) are weaker than for a purely

cohesive soil, namely

$$\varphi \leq \beta \leq \pi - \varphi \quad . \quad (6.14)$$

The proofs of the statements to appear hereafter can be found in the results given in [36] : they rely on the kinematic dual formulation of the "global equilibrium check" static analysis and can be generalized whatever the slope of the cut.

The idea would be to profit by the liberty left by condition (6.14) to perform the kinematic approach using more general piecewise rigid body motion velocity fields with any cross section for the cylindrical volume in motion. Such a rigid motion will be either a rotation around any axis  $Iz$  or a translation with any velocity  $V$  (Fig. 19).

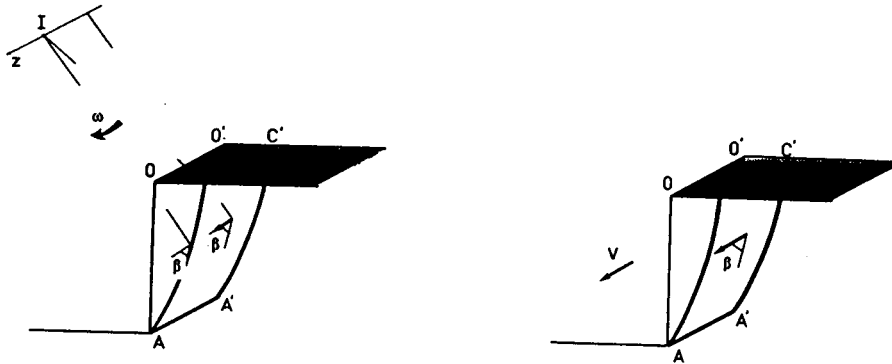


Figure 19 - Stability of a vertical cut : more general kinematic approach using piecewise rigid body motion velocity fields.

From the static analysis of view this would amount to the "global equilibrium check" of a cylindrical volume (1) = OACC'A'O', with any cross section, being considered thoroughly.

For the homogeneous vertical cut the results are as follows.

Starting from the kinematic approach it comes out that :

- for a given axis  $Iz$  defining a rotation with a velocity  $\omega$  (Fig. 20) the *most critical volume* OACC'A'O', that is the volume leading to the strongest constraint on  $(\gamma h/C)^+$ , is obtained when AC is an arc of a " $\varphi$ " logspiral with pole I, so that the induced velocity discontinuity be inclined at the angle  $\beta = \varphi$  to the surface ACC'A' at each point ; such a " $\varphi$ " logspiral is defined in polar coordinates  $(\rho, \theta)$  attached to I by the expression  $\rho = \rho_0 \exp(-\theta \tan \varphi)$  ;

- as a particular case, when a translation is considered parallel to velocity  $V$  the *most critical volume*  $OABB'A'O'$  is obtained when  $OAB$  is a triangle and  $V$  makes the angle  $\varphi$  with  $AB$  in the outwards direction (Fig. 20).

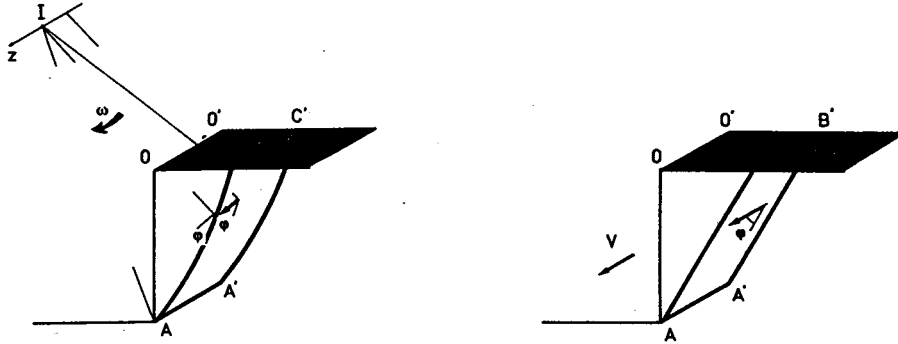


Figure 20 - Stability of a vertical cut : most critical volumes for the kinematic approaches by piecewise rigid body motion velocity fields.

Consequently, from the "global equilibrium check" point of view it follows that :

- among all cylindrical volume (1) with any cross section being tested from the "global equilibrium check" standpoint, the most critical are those with a cross section defined by " $\varphi$ " logspiral arc  $AC$  or a straight line  $AB$  (Fig. 21) ;

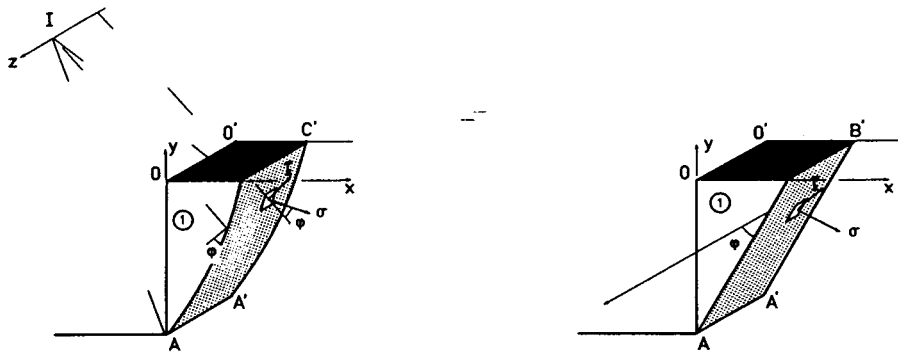


Figure 21 - Stability of the vertical cut : most critical volume for the kinematic "global equilibrium check".

- for the first type, the significant equilibrium equation which leads to the necessary constraint on  $(\gamma h/C)$  is the moment equation with respect to  $I_z$  where  $I$  is the

pole of the considered logspiral ;

- for the second type, the significant equilibrium equation is the resulting force equation in the direction inclined at the outward angle  $\varphi$  to AB in the plane Oxy (Fig. 21).

The best result obtained through these equivalent analyses turns out approximately to fit the formula [19] :

$$(\gamma h/C)^+ \leq 3.83 \tan(\pi/4 + \varphi/2) \quad (6.15)$$

(d) the difference between Eq. (5.32) and (5.33) on one side, and Ineq. (5.34) and (5.35) on the other, have led to conclusions concerning the general conditions to be fulfilled by relevant virtual k.a. velocity fields which might seem antinomic and make it difficult to imagine the possibility of passing continuously from one case to the other when  $\varphi \rightarrow 0$ .

This can be explained for instance for the relevant velocity discontinuities. For a frictional and cohesive soil ( $\varphi \neq 0$ ,  $C \neq 0$ ) it has been explained that the relevant  $[\hat{U}]$  must belong to the cone drawn in Figure 22. Making  $\varphi$  tend to zero, the allowable cone for  $[\hat{U}]$  becomes wider and flattens down into the upper half space (Fig. 15). At the same time it is seen from Eq. (5.19) where  $C$  is a constant that, keeping  $||[\hat{U}]||$  constant, the value of  $\pi(n, [\hat{U}])$  tends to infinity when  $\varphi \rightarrow 0$  unless  $[\hat{U}]$  belongs to the very boundary of the cone. It follows that,  $\varphi$  tending to zero, the relevant  $[\hat{U}]$  will tend to be confined in the velocity jump surface.

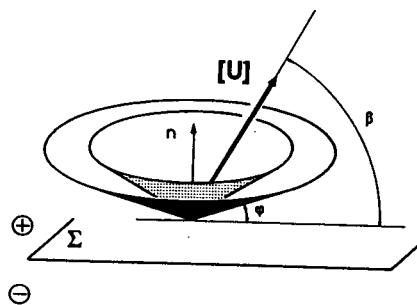


Figure 22 - The relevant velocity discontinuities as  $\varphi \rightarrow 0$ .

### 6.3.- *Comments.*

The stability analysis of the vertical cut has been chosen in this presentation with the purpose of giving an illustrative example of the kinematic approach being the dual formulation of the static one. This point of view is quite obvious in the comparison between Eq. (6.12) and Eq. (4.8) for instance.

Consequently, since only rigid motion virtual velocity fields have been used, this example is not the best suited to enhance the possibilities of the kinematic approach from outside, whose efficiency results from the variety of relevant k.a. virtual velocity fields which can be used : they make it possible and easy, through the dual formulation, to check the compatibility between equilibrium and resistance in a more elaborate way than the mere *global* equilibrium of one or a few volumes. As a matter of fact the upper bound (Ineq. 6.6) obtained in [60] is derived from such a kinematic approach where the virtual velocity field combines a rigid body motion velocity field and a shear velocity field. A good example to illustrate this point of view would be the indentation of a half space by a rigid plate (sketched in Fig. 12 and 15) with the classical results recalled in [36], and more recent developments, such as those presented in [61] in the case of an excentrated and inclined load on the plate acting on a purely cohesive soil with no resistance to tensile stresses.

## 7.- CONCLUDING REMARKS.

### 7.1.- *The fundamental principles.*

Concluding this survey of the Theory of Yield Design, the fundamental principles will first be recalled. The yield design theory relies on the necessary stability condition that "Equilibrium and Resistance should be Compatible". From this follows the *static approach*. Two papers by Hill [62, 63] may be referred to as early application of this concept.

The so called "rigid-perfectly plastic model", which has unfortunately been connected with this theory, is but an artefact without any physical reality nor theoretical or practical utility ; its only effect has been confusing ideas and giving a restrictive perspective of the possible field of applications of the yield design concept.

The *kinematic approach* is but the mathematical dualization of the static one. The involved velocity fields are *virtual* velocity fields. They do not claim any actuality as regards the true collapse modes of the considered system, even though experience seems



to show some similarity between observed collapse modes and "good" relevant k.a. virtual velocity fields.

### **7.2.- Possible fields of applications in civil engineering.**

The short historical review at the beginning of this paper already gave a list of various applications of the theory. A few words may now be added regarding recent or possibly future developments in soil mechanics, taking advantage of the versatility of the method :

stability analyses of seabed soils [64, 65], of tunnels [66, 67] ;

stability analyses assuming T or zero tension cut off strength criteria (para. 5.4.6 and 5.4.7) as first illustrated in the paper by Drucker [68], (e.g. [61]) ;

reinforced soil structures stability analyses through methods combining the yield design and the homogenization theories [33-35], or through mixed modelling, that is adopting a continuum mechanics model for the constitutive soil and a beam (or a strip) model for the reinforcing inclusions : the interaction of the two models is mechanically consistent and the strength criteria and  $\pi$  functions are available ; the sound mechanical bases make a clean sweep of all difficulties usually encountered in trying to adapt "traditional" stability analysis methods to these new circumstances.

Moreover, as already announced in [59], the same sound mechanical bases of the theory make it the reliable way to build stability analysis methods and computer codes which thoroughly comply with both the spirit and the letter of the Ultimate Limit State Design (ULSD).

As a matter of fact the approach of safety which is conveyed in ULSD requires a clear distinction to be made between the given *loads* on the considered structure and the *resistances* which may be mobilized in the constituent materials (soil, reinforcement, interfaces,...). It then introduces partial safety factors to be applied : as multipliers on the loads, greater than unity when the concerned load is unfavourable to stability, lower than unity when it is favourable, they provide the design values of the loads ; as dividers on the strength parameters, greater than unity, they provide the design values of the strength parameters.

"According to the principles of Limit States Design, the design criterion is simply to design for equilibrium in the design limit state of failure. The design criterion could

be expressed in the following way :

$$R_d \geq S_d \quad (7.1)$$

$S_d$  is the design load effect... The design resistance effect  $R_d$ ..." [69].

In order to take the adequacy of the design method into account, the design value of ratio  $R_d/S_d$  in Ineq. (6.16), denoted

$$\Gamma_s = R_d/S_d \quad (7.2)$$

and called "Method coefficient", will not be compared to unity but to a reference value, depending on the method used for the assessment of stability, which is usually greater than unity.

This short presentation of ULSD clearly shows how the Yield Design Theory is fully suited to this approach of safety which appeared as early as 1927 in Denmark and was fostered by Brinch Hansen [70].

Contrary to many traditional methods used for stability analyses (e.g. the method of slices,...) it does make the required clear distinction between loads and resistances, in the sense that the calculation of  $R_d$  on the "strength" side of Ineq. (7.1) does not, in any way, involve the loads which only appear in  $S_d$  on the "load" side of Ineq. (7.1) .

It will be easily recognized that the fundamental principle of ULSD, as quoted from [69] and expressed by Ineq. (7.1) , does not differ from the fundamental principle of the Yield Design Theory expressed in Formula (3.8), once the loads and the strength parameters have been modified through their respective partial safety factors.

Ineq. (5.4 or 5.6) and Ineq. (7.1) are not only similar but actually identical, which means that the kinematic approach clearly and rigorously results in the upper bound approach of  $\Gamma_s$  .

The reproach has sometimes been made to ULSD that, since all quantities are modified through partial safety factors, the design "mechanisms" might (or will) have nothing in common with those which may actually occur in case of collapse. Though questionable, for it may apply as well to the global safety factor approach, this remark may be partially answered to as follows. It turns out that calculation methods and computer codes based upon the Yield Design Theory are very efficient in the sense that they prove to be usually less time consuming than many traditional methods ; added to

their rigorous mechanical basis, this makes it possible to perform parametric studies which are certainly the best way to investigate the behaviour of a structure.

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