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A comprehensive stability analysis of soil nailed structures

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ABSTRACT. – The stability analysis of earth retaining structures reinforced by nails can be properly investigated through the use of the yield design theory. The theory gives a consistent mechanical framework, making it possible to deal simultaneously with the constitutive soil, modelled as a classical continuous medium, and with the reinforcing nails modelled as beams. The way the shear- and bending resistances of the nails must be taken into account in design calculations is thus clearly assessed. The magnitude of the contribution of those resistances to the stability of the structures under consideration can then be examined. The kinematic approach of the yield design theory appears, as usual, to be the most convenient for that purpose. It is implemented by using failure mechanisms with a velocity discontinuity surface which are the common tools of such analyses, but also by introducing other failure patterns where both the soil and *the nails* undergo continuous deformation within a shear zone of finite thickness. Looking for the lowest estimate so obtained for the stability factor of the structure shows that the second kind of mechanisms, which favour flexional deformation modes of the nails yields significantly better results than the first one where shear deformation modes of the nails are only acting. It is also proved that neglecting the shear – and bending resistances of the nails always results in a conservative approach compared with the complete analysis, and that the difference between the values so obtained remains slight in most cases encountered in common practice. The latter result is quite significant from the geotechnical engineering standpoint since most design calculations are actually performed under the simplifying assumption that the reinforcing nails exhibit resistance only to axial tensile forces.

1. Yield design methods applied to stability analyses

The engineering design methods specifically developed for nailed soil structures are mostly derived from the classical stability analyses of homogeneous slopes by means of failure surfaces. They must face the necessity of accounting for the strength developed by the nails across such surfaces. As a first approach, it is often acknowledged that the nails display no significant resistance apart from that due to axial tensile forces, thus assuming they behave like perfectly flexible reinforcements with no resistance to compressive force, shearing or bending. While, for example, metallic strips in the reinforced earth technique or geotextile sheets do fall into this category, such a simplifying assumption may seem questionable for soil nailing reinforcements.

However, trying to take the shear and bending strengths of nails into account leads to the problem of evaluating their specific contribution to the stability of a structure as a whole. This question has recently prompted a debate in the technical literature in order

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to assess how far this contribution could be neglected with respect to that due to axial forces. Schlosser [1983], Blondeau *et al.* [1984], then lately Jewell and Pedley [1990], Schlosser [1991], Jewell [1991], have dealt with the problem by closely examining the interactions between the nail and the surrounding soil on both sides of the failure surface, under the assumption of an elastic or elastic-plastic behaviour of the constituent materials. Nevertheless, it should be pointed out that up to now no systematic investigation of the influence of shear and bending strengths on the results of stability calculations of structures has been carried out, so that it remains difficult to assess the validity of the somewhat different conclusions drawn by the above mentioned authors.

The yield design theory (Salençon, 1983, 1990 *a*) applied to geotechnical problems stands as a mechanically consistent framework for handling such a question: it is only derived from the expression of the necessary compatibility between the equilibrium equations of the structure under consideration (from the statics standpoint) and the strength requirements of its constituent materials (soil, inclusions) and does not rely on a determination of the internal forces actually developed in the structure based upon complementary assumptions, as is the case for most "classical" approaches (e. g. the "method of slices").

In this paper the presentation starts with a with a simple case where the static reasoning "from outside" is applied to the global equilibrium of triangular-shaped volumes of reinforced soil described through the "mixed modelling" of the soil as a continuum and of the nails as beams. This analysis is proved to be equivalent to the kinematic approach "from outside" of the theory of yield design, applied with rigid body translation motion failure mechanisms, and yields upper bound estimates for the non dimensional factor which governs the stability of the structure.

The kinematic approach is then generalized to rotational failure mechanisms involving either velocity discontinuity surfaces, or deformation zones where the velocity of both the soil and the nails remain continuous. Attention is particularly focused on this second type of failure mechanisms which favours a flexional deformation mode of the nails and therefore refers to their bending strength characteristics instead of their shearing resistance as in the case of velocity discontinuities. They are shown to yield better estimates for the stability factor of the structure and lead to the conclusion that only a small, often negligible, proportion of the shear strength of the nails is actually mobilized and should therefore be taken into account in stability analyses.

2. Stability analysis of a structure reinforced by nails

2.1. Statement of the problem

Consider for illustrative purposes the example of a vertical cut in a homogeneous soil reinforced by one single layer of infinitely long nails, regularly spaced in a horizontal plane (*Fig.* 1).

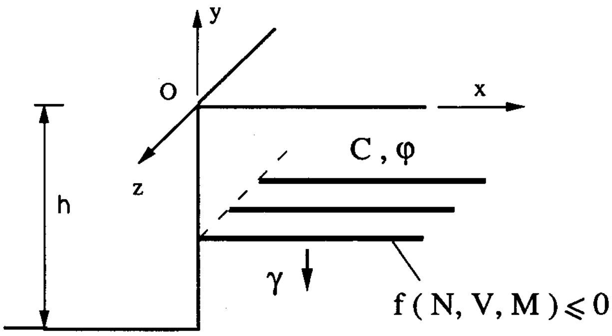

Fig. 1. – Stability analysis of a reinforced vertical cut.

According to the yield design theory, performing the stability analysis of such a structure requires three sets of data:

- the geometry of the structure, *i.e.* the depth h of the excavation,
- its loading conditions, reduced here to the specific weight γ of the soil, since it will be assumed that no surcharge is applied,
- and finally the strength properties of the constituent materials.

A mixed modelling of the structure will be adopted for this study.

The soil is modelled as a classical 3 D continuum where the stress tensor is denoted by $\underline{\underline{\sigma}}$ (tensile normal stresses positive). The stress vector upon any material surface element of outward unit normal $\underline{n}$ within the soil is given by $\underline{T}(\underline{n})=\underline{\underline{\sigma}}.\underline{n}$ (*Fig.* 2-*a*).

The reinforcing nails are modelled as beams, 1 D continuous media. A positive orientation along any nail is chosen, defining the unit tangent vector $\underline{e}_n$ and the right handed frame $(\underline{e}_n, \underline{e}_v, \underline{e}_z)$ at each point P. It is assumed that the cross section of the nail at point P admits $\underline{e}_v$ and $\underline{e}_z$ as symmetry axes. Since it will be stated later on that the problem may be studied as a two dimensional one, the internal efforts to be considered in the beam reduce to the axial force N, the shear force V and the bending moment M which are the components of the resultant force and moment acting at point P on the "upstream" part of the nail due to the "downstream" part (*Fig.* 2-*b*).

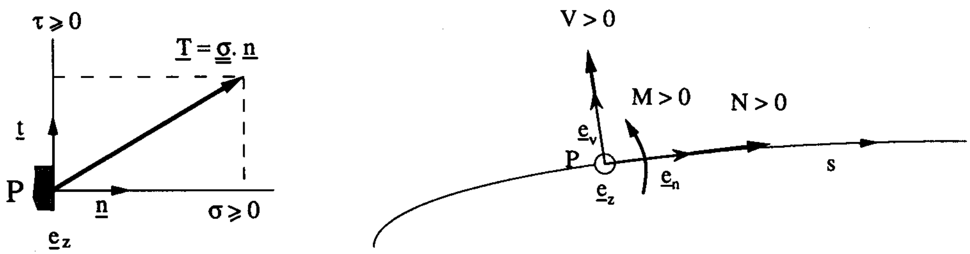

Fig. 2. – Sign convention for the internal forces:
a) within the soil. *b*) along the nails.

The strength of the homogeneous constituent soil is defined by Coulomb's isotropic strength condition with a cohesion C and a friction angle φ:

$$(1) \qquad f^s(\underline{\underline{\sigma}}) = \operatorname*{Sup}_{i,j} \{ \sigma_i (1+\sin\varphi) - \sigma_j (1-\sin\varphi) - 2\,C\cos\varphi \} \leqq 0$$

(σ_i, σ_j principal stresses).

Due to the very large dimension of the considered structure along Oz, to the regular spacing of the nails along Oz, to the uniform loading conditions, to the homogeneity and isotropy of the soil and to the symmetry hypothesis regarding the cross section of the nails, it may be shown that the 3 D problem can be conveniently dealt with in the Oxy plane as a dimensional plane strain yield design problem (*cf.* [Salençon, 1983] for the rigourous definition of this intuitive concept).

It follows that the strength criterion for the nails needs only to refer to the above defined N, V, M considered from now on as forces and moment per unit transverse length along Oz. The following expression may be adopted (Anthoine, 1987):

$$(2) \qquad f(N, V, M) = (N/N_0)^2 + (V/V_0)^2 + |M/M_0| - 1 \leqq 0$$

where N_0, V_0, M_0 denote the ultimate values of N, V and M.

It is worth pointing out that such a formula, which proves slightly conservative when compared with those proposed by several authors [Neal, 1961] [Sobotka, 1954, 1955] is far easier to handle.

For simplicity's sake, perfect bonding will be assumed between the soil and the inclusions.

Generally speaking, for this structure to be safe, it is necessary to exhibit a system of internal forces (stress field $\underline{\underline{\sigma}}$ within the bulk of soil, distributions of N, V and M along the nails) in equilibrium with the loading (weight of the soil) while complying with the strength conditions of the constituents at every point. It can be shown from dimensional analysis arguments that the stability of the structure is governed by the dimensionless parameters $\gamma h/C$, N_0/Ch, V_0/N_0, $M_0/N_0 h$ and φ, so that the following equivalence may be written:

$$(3) \qquad \text{Stability of the structure} \Leftrightarrow \gamma h/C \leqq K^*$$

where K^* appears as a function of N_0/Ch, V_0/N_0, $M_0/N_0 h$ and φ.

2.2. Static analysis of triangular volumes

Referring to Coulomb's original reasoning in the case of a homogeneous soil (which is extensively analysed in [Salençon, 1990-*a*]), the overall equilibrium of a volume of reinforced soil bounded by any arbitrary line AB passing through the toe and inclined at an angle α with respect to the vertical direction (*Fig.* 3) will now be considered.

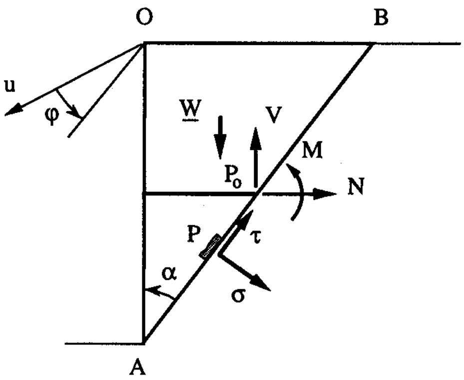

Fig. 3. – Global equilibrium
of a triangular volume of reinforced soil.

It clearly appears that a *necessary* condition for the structure to be safe, is that the *global equilibrium* of any such volume OAB be possible under its own weight and the resisting forces developed by the soil and the nails along AB, on account of their respective strength capacities; that is:

(4) $$\gamma h/C \leqq K^* \\ \Downarrow$$

Equilibrium of volume OAB when subjected to:

– its weight $\underline{W}$,

– the stress distribution (σ, τ) along AB,

– the solicitations (N, V, M) developed by the layer of nails at its intersecting point with AB

under the following strength conditions derived from Ineq. (1) and (2):

(5-*a*) $$|\tau| \leqq C - \sigma \tan \varphi$$

(5-*b*) $$f(N, V, M) \leqq 0.$$

Expressing the resultant equilibrium of the volume projected onto the O*u*-axis inclined at angle φ with AB, it takes the form

(6) $$W \cos(\alpha+\varphi) = \int_{AB} (\sigma \sin\varphi + \tau \cos\varphi)\, ds + N \sin(\alpha+\varphi) + V \cos(\alpha+\varphi)$$

with $W = (1/2)\gamma h^2 \tan \alpha$.

Now, at any point P of AB

$$\sigma \sin\varphi + \tau \cos\varphi \leqq \sup_{(\sigma,\tau)} \{ \sigma \sin\varphi + \tau \cos\varphi ; |\tau| \leqq C - \sigma \tan\varphi \}$$

that is:

(7) $$\sigma \sin\varphi + \tau \cos\varphi \leqq C \cos\varphi.$$

The latter inequality can be given a geometrical interpretation from Figure 4. It results from the fact that at any point P of AB the algebraic value (equal to $-(\sigma \sin \varphi + \tau \cos \varphi)$) of the projection onto the oriented axis O*u* (or P*u*) of the stress vector $\underline{T}$ is bounded by $-C \cos \varphi$, due to the limitation imposed on $\underline{T}$ by the strength criterion of the soil. Such a limitation is classically expressed in the Mohr plane (σ, τ) through a convex domain, the boundary of which (called the intrinsic curve) is made up of two semi-lines inclined at angle φ with the σ-axis.

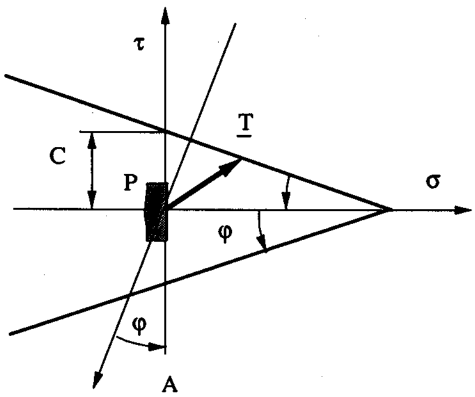

Fig. 4. – Limitation imposed by the Coulomb strength condition on the resisting contribution of the soil.

Likewise, the strength condition (5-*b*) expressed in the form of inequality (2) allows us to derive an upper bound for the last two terms of Eq. (6), without resorting to any further assumption concerning N and/or V:

$$N \sin(\alpha+\varphi) + V \cos(\alpha+\varphi) \leqq \sup_{(N, V)} \{ N \sin(\alpha+\varphi) + V \cos(\alpha+\varphi); f(N, V, M) \leqq 0 \}$$

that is:

$$\text{(8)} \qquad N \sin(\alpha+\varphi) + V \cos(\alpha+\varphi) \leqq [N_0^2 \sin^2(\alpha+\varphi) + V_0^2 \cos^2(\alpha+\varphi)]^{1/2}$$

The geometrical interpretation of this inequality is quite similar to that given for inequality (7). The strength condition (2) implies that the force of components (N, V) which is developed by the nail is always bounded by the *elliptical* domain of equation $(N/N_0)^2 + (V/V_0)^2 \leqq 1$, obtained in the most favourable case, that is when the moment M reduces to zero. The right hand member of (8) is but the maximum value PH of the projection onto the P*u*-axis (counted positive in the sense opposite to *u*) of the resisting force mobilized by the nail, on account of its strength capacities, as illustrated in Figure 5.

Combining the equilibrium equation (6) with the limitations imposed by inequality (7) and (8) on the resisting constructions of the soil and the inclusion respectively yields:

$$\text{(9)} \qquad W \cos(\alpha+\varphi) \leqq \frac{h}{\cos \alpha} C \cos \varphi + [N_0^2 \sin^2(\alpha+\varphi) + V_0^2 \cos^2(\alpha+\varphi)]^{1/2}.$$

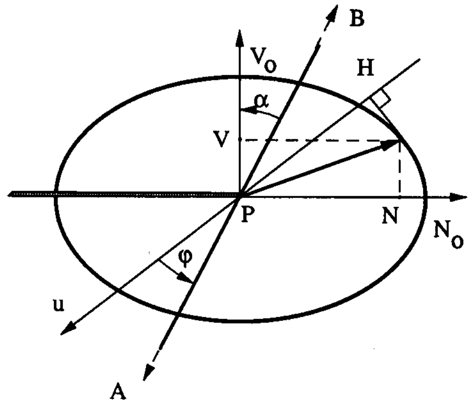

Fig. 5. — Limitation induced by the strength criterion of the nail on its resisting force mobilized at angle φ with AB.

It should be emphasized that, relying exclusively upon the datum of strength conditions, it has been possible to determine the *maximum resisting contribution* in the equilibrium equation (6) available from both the soil and the reinforcements, thereby leading to the above inequality. Assuming $0<\alpha<\pi/2-\varphi$, inequality (9) becomes:

$$\text{(9-}a\text{)} \qquad \gamma h/C \leqq 2 \cos \varphi/(\sin \alpha \cos (\alpha+\varphi))+4r (\tan \alpha)^{-1} [v^2+\tan^2 (\alpha+\varphi)]^{1/2}$$

with $v=N_0/V_0$ and $r=N_0/2Ch$.

The meaning of this inequality is obvious: derived from the global equilibrium check of volume OAB on account of the strength criteria, it expresses a necessary condition for the structure to be safe. Otherwise stated, it means that the structure is *certainly unsafe* as soon as the actual value of the non dimensional factor $\gamma h/C$ exceeds the value of the right hand member of inequality (9-*a*).

Consequently, inequality (9-*a*) provides, for any value of α ranging between 0 and $\pi/2-\varphi$, an *upper bound estimate* for the critical value K^* of the stability factor. It can be noticed that for the particular value $\alpha=\pi/4-\varphi/2$, one obtains:

$$\text{(10)} \qquad K^* \leqq 4 K_p^{1/2} [1+r(v^2+K_p)^{1/2}]$$

where $K_p=\tan^2((\pi/4)+(\varphi/2))$ is the passive earth pressure coefficient, reducing to the classical result in the non-reinforced case ($r=0$):

$$K^* \leqq 4 \tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right).$$

Minimizing the right hand member of (9-*a*) with respect to α ($0<\alpha<\pi/2-\varphi$) makes it possible to determine, for any set of parameters r and v characterizing the reinforcement, the best possible upper bound estimate of K^* to be expected from such an analysis.

2.3. COMMENTS

2.3.1. The same kind of reasoning would apply when projecting the resultant equilibrium condition onto any Ou-axis inclined at angle ψ with AB. The corresponding equation becomes:

$$W \cos(\alpha+\psi)=\int_{AB} (\sigma \sin \psi+\tau \cos \psi)\, ds+N \sin(\alpha+\psi)+V \cos(\alpha+\psi)$$

where upper bounds for the different terms of the right hand member can again be derived. For the last two terms corresponding to the contribution of the nails, formula (8), where φ is to replaced by ψ, still applies, whereas one may write for the expression placed under the integral sign:

$$\sigma \sin \psi+\tau \cos \psi \leqq \sup_{(\sigma,\, \tau)} \{ \sigma \sin \psi+\tau \cos \psi ; |\tau| \leqq C-\sigma \tan \varphi \}$$

which can be written explicitly as

$$\sigma \sin \psi+\tau \cos \psi \leqq \begin{cases} C(\tan \varphi)^{-1}.\sin \psi & \text{if} \quad \varphi \leqq \psi \leqq \pi-\varphi \\ +\infty & \text{otherwise,} \end{cases}$$

as can be seen from Figure 4.

Assuming $\varphi \leqq \psi < \pi/2$ and $0<\alpha<\pi/2-\varphi$, the following inequality is eventually obtained:

$$\gamma h/C \leqq K(\alpha, \psi) \quad \text{whence} \quad K^* \leqq K(\alpha, \psi)$$

with

$$K(\alpha, \psi)=2(\tan \varphi)^{-1} \sin \psi/(\sin \alpha.\cos(\alpha+\psi))+4r(\tan \alpha)^{-1}[v^2+\tan^2(\alpha+\psi)]^{1/2}.$$

Minimizing the upper bound estimate $K(\alpha, \psi)$ with respect to angular parameters α and ψ :

$$K^* \leqq \underset{(\alpha,\, \psi)}{\text{Min}} \{ K(\alpha, \psi) ; \varphi \leqq \psi < \pi/2,\ 0<\alpha<\pi/2-\psi \}$$

would lead to an improvement of the upperbound estimate given by inequality (9-*a*).

2.3.2. In order to determine the resisting forces mobilized in the nails as they are to be taken into account in "classical" stability analysis methods, some authors ([Schlosser, 1983], [Blondeau *et al.*, 1984]) refer to a "maximum work principle". It follows from the preceding analysis that in actual fact the only relevant problem consists in searching for the limitation imposed by the strength capacities on the contribution due to the resisting forces developed by the reinforcements. Solving this problem only relies on equilibrium considerations involving a maximization procedure with constraints which turns out to be of the same kind as that used for determining the maximum resisting contribution of the soil.

3. Yield design kinematic approach to the problem

3.1. Fundamentals of the kinematic method

The yield design kinematic approach proceeds from the mathematical dualization of the static reasoning by means of the *principle of virtual work* [Salençon, 1983]. The principle first requires a definition of the virtual motions to be considered in the mechanical description of the system. Within the context of a mixed modelling for the reinforced soil, such a virtual motion is defined in the following way.

- The virtual motion of the soil, modelled as a 2D continuum, is described by a vector field $\underline{U}$ throughout its volume $\mathscr{V}$.

- The construction of the virtual motions of the nail, modelled as a beam loaded within the plane of the figure, consists in assigning a couple of independent vectors $(\underline{U}(s), \underline{\Omega}(s))$ to any point P(s) along the line $\mathscr{L}$. $\underline{U}(s)$ is defined by continuity with the virtual motion in the soil and represents the virtual velocity of the "neutral axis" of the nail at point P(s), while $\underline{\Omega}(s)=\Omega(s)\underline{e}_z$ is the virtual rotation of the "transverse section" at the same point (*Fig.* 6).

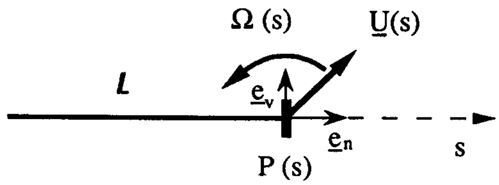

Fig. 6. — Virtual motions of the nail schematized as a beam.

The *principle of virtual work* states that, given any system of internal forces (stress field $\underline{\underline{\sigma}}$ in $\mathscr{V}$, distribution of N, V, M along $\mathscr{L}$) in equilibrium with the loading of the structure, the following equality is satisfied

$$\mathscr{W}_e(\{\underline{U}\})+\mathscr{W}_i(\{\underline{U}\})=0 \quad \text{whatever } \{\underline{U}\} \tag{11}$$

where $\{\underline{U}\}$ denotes any virtual motion as previously defined.

$\mathscr{W}_e(\{\underline{U}\})$ is the virtual work of the *external forces*, that is, in the present case, the work of the gravity forces

$$\mathscr{W}_e(\{\underline{U}\})=\int_{\mathscr{V}} -\gamma\,\underline{U}\,.\,\underline{e}_y\,dx\,dy \tag{12}$$

and $\mathscr{W}_i(\{\underline{U}\})$ represents the virtual work of the *internal forces*. Its expression is:

$$\mathscr{W}_i(\{\underline{U}\})=-\int_v (\underline{\underline{\sigma}} : \underline{\underline{d}})\,d\mathscr{V}-\int_{\mathscr{L}} w_i(s)\,ds \tag{13}$$

where $\underline{\underline{d}}=1/2\ (\underline{\underline{\text{grad}}}\ \text{U}+{}^t\underline{\underline{\text{grad}}}\ \text{U})$ is the strain rate field associated with $\underline{\text{U}}$, and [Salençon, 1988]:

$$(14)\qquad w_i(s)=\text{N}(s)\frac{d\underline{\text{U}}(s)}{ds}.\underline{e}_n+\text{V}(s)\left(\frac{d\underline{\text{U}}(s)}{ds}.\underline{e}_v-\Omega(s)\right)+\text{M}(s)\frac{d\Omega(s)}{ds}.$$

$\underline{\text{U}}$ and/or Ω may be discontinuous across surfaces Σ in the soil and/or across several points $\text{P}_k(s_k)$ along $\mathscr{L}$. In such a case, one must add to the right hand side of Eq. (13) the discontinuity terms:

$$(15)\qquad -\int_\Sigma(\underline{\underline{\sigma}}.\underline{n})[\![\underline{\text{U}}]\!]\,d\Sigma$$

for the soil and

$$(16)\qquad \sum_k(\text{N}(s_k)[\![\underline{\text{U}}(s_k)]\!].\underline{e}_n+\text{V}(s_k)[\![\underline{\text{U}}(s_k)]\!].\underline{e}_v+\text{M}(s_k)[\![\Omega(s_k)]\!])$$

for the nail where $[\![\quad]\!]$ denotes the jump across Σ following its unit normal $\underline{n}$ or across point P following $\underline{e}_n$.

Introducing then:

$$(17)\qquad \pi(\underline{\underline{d}})=\underset{\underline{\underline{\sigma}}}{\text{Sup}}\{\underline{\underline{\sigma}}:\underline{\underline{d}};f^s(\underline{\underline{\sigma}})\leqq 0\}$$

where $f^s(\underline{\underline{\sigma}})$ is the Coulomb strength condition of the soil introduced in Eq. (11), and

$$(18)\qquad \pi\left(\frac{d\underline{\text{U}}}{ds},\frac{d\Omega}{ds},\Omega\right)=\underset{(\text{N, V, M})}{\text{Sup}}\{w_i(s);f(\text{N, V, M})\leqq 0\}$$

where function f is defined by Eq. (2), it turns out that a necessary condition for the structure to be safe is given by

$$(19)\qquad \mathscr{W}_e(\{\underline{\text{U}}\})\leqq\mathscr{W}_r(\{\underline{\text{U}}\})\quad\text{whatever }\{\underline{\text{U}}\}$$

with:

$$(20)\qquad \mathscr{W}_r(\{\underline{\text{U}}\})=\int_{\mathscr{V}}\pi(\underline{\underline{d}})\,d\mathscr{V}+\int_{\mathscr{L}}\pi\left(\frac{d\underline{\text{U}}}{ds},\frac{d\Omega}{ds},\Omega\right)ds.$$

The functional $\mathscr{W}_r(\{\underline{\text{U}}\})$ is called the *maximum resisting work* in the virtual motion $\{\underline{\text{U}}\}$. In the case of discontinuities of $\underline{\text{U}}$ and Ω the following additional terms must be introduced:

$$(21)\qquad \int_\Sigma\pi(\underline{n};[\![\underline{\text{U}}]\!])\,d\Sigma+\sum_k\pi_k([\![\underline{\text{U}}(s_k)]\!],[\![\Omega(s_k)]\!])$$

with

$$\pi(\underline{n};\ [\![\underline{U}]\!]) = \sup_{\underline{\underline{\sigma}}}\left\{(\underline{\underline{\sigma}}.\underline{n}).[\![\underline{U}]\!];\ f^s(\underline{\underline{\sigma}}) \leqq 0\right\} \tag{22}$$

and

$$\pi_k([\![\underline{U}(s_k)]\!],\ [\![\Omega(s_k)]\!]) = \sup_{(N,\,V,\,M)}\left\{N[\![\underline{U}(s_k)]\!].\underline{e}_n + V[\![\underline{U}(s_k)]\!].\underline{e}_v + M[\![\Omega(s_k)]\!];\ f(N, V, M) \leqq 0\right\}. \tag{23}$$

The implementation of the yield design kinematic approach relies upon the fundamental inequality (19). The virtual motions $\{\underline{U}\}$ will be called "failure mechanisms" of the reinforced soil structure.

3.2. Kinematic approach using uniform translation motions

Consider a failure mechanism in which the same triangular volume OAB as that defined in 2.2 is given a downwards uniform translation motion with the velocity $\underline{U}$ inclined at angle φ with respect to AB, while the remaining part of the structure is kept motionless (*Fig.* 7).

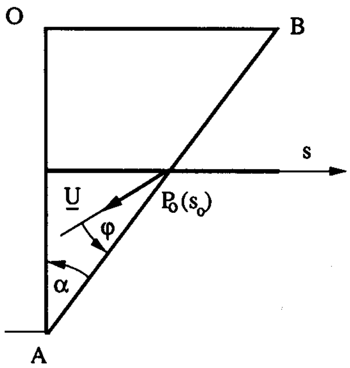

Fig. 7. – Virtual translation motion of OAB.

The work performed by the external forces (gravity forces) in such a failure mechanism is easily derived from Eq. (12), as

$$\mathscr{W}_e(\{\underline{U}\}) = WU\cos(\alpha+\varphi) \qquad (U = |\underline{U}|). \tag{24}$$

On the other hand, the maximum resisting work is equal to the sum of two terms.

The first term, relating to the resisting contribution of the soil, reduces to

$$\mathscr{W}_r^{soil}(\{\underline{U}\}) = \int_{AB} \pi(\underline{n};\ [\![\underline{U}]\!])\,dl$$

since no (virtual) deformation occurs outside the discontinuity line AB, where $[\![\underline{U}]\!] = \underline{U}$.

Hence from Eq. (21)

(25) $$\mathscr{W}_r^{\text{soil}}(\{\underline{U}\}) = CU \frac{h}{\cos\alpha} \cos\varphi.$$

The second term represents the maximum resisting work developed by the nail in the failure mechanism under consideration. Denoting by $P_0(s_0)$ the intersecting point between AB and the nail, and having oriented the latter from the left hand side to the right hand

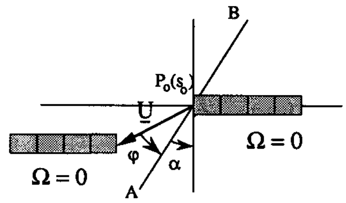

Fig. 8. – Virtual motion of the nail induced by a uniform translation motion of volume OAB.

side of the figure, the virtual motion of the nail is defined as follows (*Fig.* 8)

$$\underline{U}(s) = \underline{U}, \quad \Omega(s) = 0 \quad \text{for } s < s_0$$
$$\underline{U}(s) = 0, \quad \Omega(s) = 0 \quad \text{for } s > s_0$$

hence

$$[\![\underline{U}(s_0)]\!] = -\underline{U} \quad \text{and} \quad [\![\Omega(s_0)]\!] = 0.$$

It follows that the maximum resisting work developed by the nail, computed from Eqs. (21) and (23), reduces to

(26) $$\mathscr{W}_r^{\text{nail}}(\{\underline{U}\}) = U \,\text{Max}\{N \sin(\alpha+\varphi) + V\cos(\alpha+\varphi); f(N, V, M) \leqq 0\}$$

or, taking into account the expression (2) of the strength condition for the nail,

(27) $$\mathscr{W}_r^{\text{nail}}(\{\underline{U}\}) = U[N_0^2 \sin^2(\alpha+\varphi) + V_0^2 \cos^2(\alpha+\varphi)]^{1/2}.$$

Putting Eqs. (24), (25) and (27) into the fundamental inequality (19) yields

$$WU\cos(\alpha+\varphi) \leqq CU\frac{h\cos\varphi}{\cos\alpha} + [N_0^2 \sin^2(\alpha+\varphi) + V_0^2\cos^2(\alpha+\varphi)]^{1/2} U$$

that is, after dividing both members by the positive factor U, the same inequation as (9). This should be no surprise since it is to be reminded that the kinematic approach is but the dualization of the static approach by means of the virtual work principle. More particularly, in the present case, implementing the kinematic approach through the use of a translating block failure mechanism is strictly equivalent to the static reasoning performed in section 2.2.

3.3. Kinematic approach with rotational failure mechanisms

The kinematic approach can be extended to considering virtual velocity fields in which a volume such as OAB (*Fig.* 9) rotates about a point F, thereby inducing a velocity jump across the boundary line AB which separates the volume from the rest of the structure.

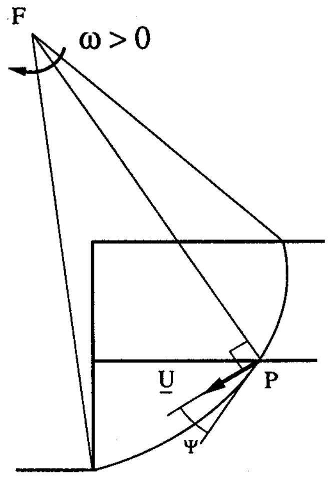

Fig. 9. — Rotational failure mechanism.

The expression of the work done by the external forces may be written in the form:

$$\mathcal{W}_e(\{\underline{U}\}) = \mathcal{M}_w . \omega \tag{28}$$

where ω represents the angular velocity of the rotating volume OAB (clockwise positive), and $\mathcal{M}_w$ the moment of the gravity forces applied to OAB with respect to F.

Likewise, it can be seen that the maximum resisting work becomes

$$\mathcal{W}_r(\{\underline{U}\}) = \mathcal{W}_r^{soil}(\{\underline{U}\}) + \mathcal{W}_r^{nail}(\{\underline{U}\}) \tag{29}$$

with

$$\mathcal{W}_r^{soil}(\{\underline{U}\}) = \mathcal{M}_r^{soil} . \omega$$

and

$$\mathcal{W}_r^{nail}(\{\underline{U}\}) = \mathcal{M}_r^{nail} . \omega$$

where $\mathcal{M}_r^{soil}$ (resp. $\mathcal{M}_r^{nail}$) denotes the *maximum resisting moment* about point F developed by the distribution of normal and shear stresses (σ, τ) acting upon OAB along AB (resp. by N, V, M in the nail at its intersecting point $P_0(s_0)$ with AB) on account of the strenth condition (5-*a*) (resp. (5-*b*)). Therefore, taking ω as positive, inequality (19) reduces to

$$\mathcal{M}_w \leqq \mathcal{M}_r^{soil} + \mathcal{M}_r^{nail}. \tag{30}$$

Once again, this result shows that the yield design kinematic approach using rotational failure mechanisms may be interpreted as the dualization through the principle of virtual work of the same kind of reasoning as that performed in section 2.2, which would consist in checking the *moment equilibrium* of volume OAB with respect to F. The computation of $\mathcal{M}_r^{soil}$ (*see* for example [Salençon, 1983] or [Salençon, 1990-*a*]) shows that for its value to remain finite, and thus to lead to a non-trivial upper bound estimate for K*, it is necessary that the angle ψ between the velocity at any point of AB and the tangent at the same point be comprised between φ and $\pi-\varphi$ (*see* also para. 2.3.*a*). In the particular case when $\psi=\varphi$, the corresponding curve AB is an arc of logspiral of angle φ and focus F. A computational software based upon the use of such logspirals has been devised for dealing with the stability analysis of reinforced retaining structures ([Anthoine and Salençon, 1989] [Anthoine, 1990] [de Buhan *et al.*, 1992]).

4. Failure mechanisms involving continuous deformation patterns of the nails

The yield design kinematic approach may be generalized to failure mechanisms in which both the soil and the nails undergo continuous deformations. For example the velocity jump line of Figure 9 may be replaced by a narrow "shear zone" inside which

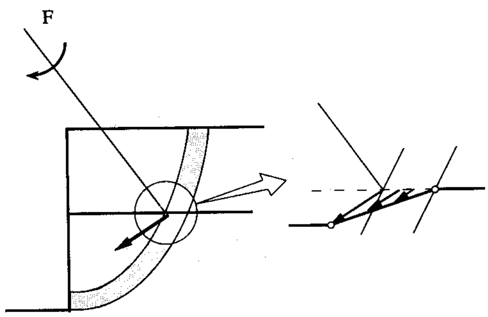

Fig. 10. – Rotational failure mechanism with "shear zone".

the velocity remains continuous as sketched in Figure 10. Such (virtual) deformation modes are suggested by the observation of (actual) failure patterns of reinforced soil structures in full-scale experiments [Plumelle, 1987] (*Fig.* 11). As it will appear below, the main difference between such failure mechanisms and those with velocity discontinuities examined in the previous section lies in the way the strength capacities of the nail are mobilized.

4.1. Strength mobilization of a nail in a "shear zone"

Let us consider for illustrative purpose, a thin layer of soil of constant thickness intersecting a nail over a distance δ (*Fig.* 12). Assuming for instance that the velocity in the soil (and therefore along the nail) varies linearly from $P_0(s_0)$ to $P_1(s_1)$, the virtual motion of the nail may be defined as follows:

$$s<s_0: \quad \underline{U}(s)=\underline{U},\ \Omega(s)=0$$

$$s_0 < s < s_1 = s_0 + \delta : \quad \underline{U}(s) = \underline{U}\frac{s_1 - s}{\delta}, \; \Omega(s) = \Omega \text{ constant}$$

$$s > s_1 : \quad \underline{U}(s) = 0, \; \Omega(s) = 0$$

whence

$$\left.\begin{aligned} \frac{d\underline{U}(s)}{ds} \cdot \underline{e}_n &= (U/\delta) \sin(\alpha + \varphi), \\ \frac{d\underline{U}(s)}{ds} \cdot \underline{e}_v - \Omega(s) &= (U/\delta) \cos(\alpha + \varphi) - \Omega, \\ \frac{d\Omega}{ds}(s) &= 0 \end{aligned}\right\} \quad \text{for} \quad s_0 < s < s_1$$

and

$$[\![\underline{U}(s_0)]\!] = [\![\underline{U}(s_1)]\!] = 0, \qquad [\![\Omega(s_0)]\!] = -[\![\Omega(s_1)]\!] = \Omega.$$

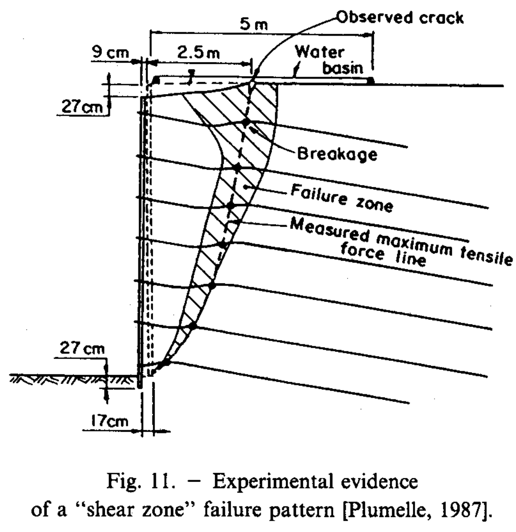

Fig. 11. – Experimental evidence of a "shear zone" failure pattern [Plumelle, 1987].

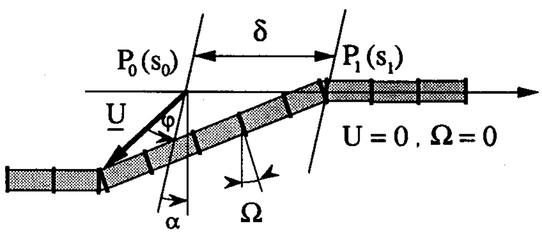

Fig. 12. – Sketch of the virtual deformation pattern of a nail in a shear zone.

The maximum resisting work developed by the nail in such a failure mechanism may be computed from Eqs. (14), (16), (18) and (23) which leads to the following results:

$$s_0 < s < s_1$$

$$\pi\left(\frac{d\underline{U}}{ds}, \frac{d\Omega}{ds}, \Omega\right) = \underset{(N, V, M)}{\text{Max}} \left\{\frac{NU}{\delta} \sin(\alpha+\varphi) + V\left(\frac{U\cos(\alpha+\varphi)}{\delta} - \Omega\right); f(N, V, M) \leqq 0\right\}$$
$$= N_0 U/\delta \left[\sin^2(\alpha+\varphi) + v^2\left(\cos(\alpha+\varphi) - \frac{\delta\Omega}{U}\right)^2\right]^{1/2}$$

where $v = V_0/N_0$; and

$$\pi_0([\![\underline{U}(s_0)]\!], [\![\Omega(s_0)]\!]) = \pi_1([\![\underline{U}(s_1)]\!], [\![\Omega(s_1)]\!]) = M_0 |\Omega|.$$

It follows that

$$\mathscr{W}_r^{nail}(\{\underline{U}\}) = N_0 U \left[\sin^2(\alpha+\varphi) + v^2\left(\cos(\alpha+\varphi) - \frac{\delta\Omega}{U}\right)^2\right]^{1/2} + 2 M_0 |\Omega|$$

or, introducing the following dimensionless parameters

$$\mu = 2 M_0/N_0 \delta \qquad \text{and} \qquad \lambda = \delta\Omega/U$$

(31) $$\mathscr{W}_r^{nail}(\{\underline{U}\}) = N_0 U (\{\sin^2(\alpha+\varphi) + v^2(\cos(\alpha+\varphi) - \lambda)^2]^{1/2} + \mu |\lambda|).$$

Fig. 13. — (*a*) $\Omega=0$. (*b*) $\Omega=(U\sin(\alpha+\varphi)/\delta)(\theta=\alpha+\varphi)$.
$\Omega=(U/\delta)\sin(\alpha+\varphi)$, $\theta=\alpha+\varphi$.

Two extreme cases are now more particularly examined.

$$(a)\ \Omega=0, \qquad (b)\ \Omega=\frac{U\sin(\alpha+\varphi)}{\delta}(\theta=\alpha+\varphi).$$

4.1.1. $\lambda=0$, that is $\Omega=0$ (*Fig.* 13-*a*). There is no virtual rotation of the transverse sections of the nail, and therefore no concentrated rotations (hinges) appear at points

P_0 and P_1. Eq. (31) yields then:

$$\mathcal{W}_r^{nail}\{\underline{U}\} = N_0\, U\,[\sin^2(\alpha+\varphi) + v^2 \cos^2(\alpha+\varphi)]^{1/2}.$$

This expression is exactly the same as the one obtained in the case of a velocity jump surface [Eq. (27)]. It shows that such mechanisms do not mobilize any resisting contribution of the nails from their resistance to bending since they do not imply any bending at all. The maximum resisting work comes only from the resistance to normal and shear forces.

4.1.2. $\lambda = \cos(\alpha+\varphi)$, that is $\Omega = U \cos(\alpha+\varphi)/\delta$ (*Fig.* 13-*b*). The virtual motion of the nail is such that the "cross sections" of the nail remain normal to its "neutral axis" (*Navier Bernoulli condition*), thus inducing *hinges* (rotation discontinuities) at points P_0 and P_1. The maximum resisting work for such "double hinge" mechanisms becomes:

$$\mathcal{W}_r^{nail}(\{\underline{U}\}) = N_0\, U\,(|\sin(\alpha+\varphi)| + \mu\,|\cos(\alpha+\varphi)|) \tag{32}$$

It could be said that the shearing resistance of the nail is not mobilized in such mechanisms since Eq. (32) displays only the normal and bending strength characteristics.

4.2. Practical consequences for the stability analysis of structures

A comprehensive study, aimed at comparing the results obtained by using either velocity discontinuity surfaces, or mechanisms involving deformation patterns of the nail complying with the Navier Bernoulli condition (case 4.1.2. of the previous paragraph) has been carried out on the example of a vertical cut made of a purely cohesive soil ($\varphi = 0$) and reinforced by a network of uniformly distributed nails [Anthoine, 1987]. In such a case, the velocity jump lines AB reduce to arcs of circles across which the velocity discontinuity remains purely tangential ("slip circles") while the deformation zones are circular rings where the soil undergoes pure shear deformation. The computations show that the maximum resisting work developed by the nails in the mechanisms of the first type (analogous to that displayed in Figure 9) is mainly due to the normal and shear resistance of the nails, the contribution of their bending resistance mobilized through the virtual rotation of volumes OAB being in most cases negligible. On the other hand, in the second type of mechanisms similar to those represented in Figure 10 the maximum resisting work of the nails can only be attributed to their normal and bending resistance.

An example taken from this study will help drawing conclusions from such an analysis. Figure 14 represents the variations of different *upper bound estimates* of the ratio $K^*/(1+r)$ as a function of the reinforcement factor defined as $r = n N_0 / 2 C h$, where n is the number of reinforcement layers.

- *Curve n° 1* corresponds to a "*slip circle*" analysis in which both shear and bending strengths are neglected ($v = V_0/N_0 = 0$, $m = M_0/N_0 h = 0$).

- *Curve n° 2* relates to the same analysis where v is set equal to 0.5 and m to 5×10^{-4}. This value of m corresponds to a 10 meter high vertical cut reinforced by circular bars of diameter $d \simeq 2.3 \times 10^{-2}$ meter.

• *Curve n° 3* refers to an analysis by failure mechanisms of the second type (circular shear zones) with the same data as for curve n° 2 ($v=0.5$; $m=5\times 10^{-4}$). It should be noted that in the latter calculation, the thickness of the shear zone is an additional parameter in the minimization process for deriving the best upperbound estimate for $K^*/(1+r)$.

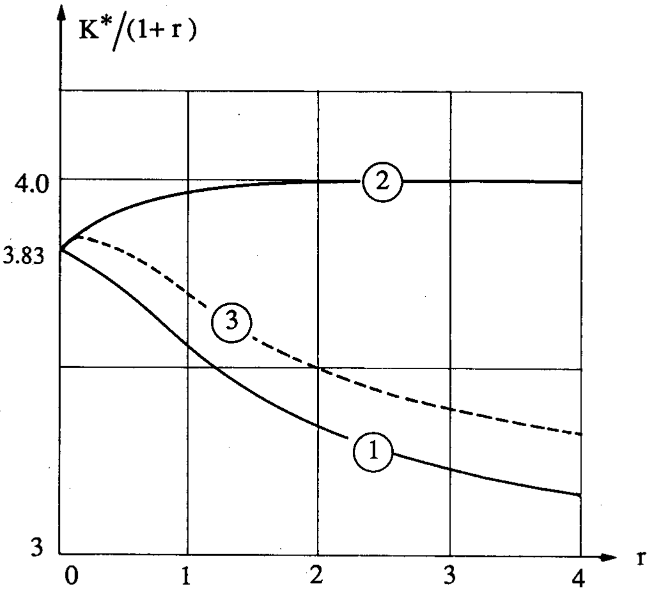

Fig. 14. – Yield design kinematic analysis of the stability of a nailed vertical cut using different kinds of failure mechanisms: 1 "slip circle" analysis ($v=0$; $m=0$); 2 "slip circle" analysis ($v=0.5$; $m=5\times 10^{-4}$); 3 analysis by mechanisms with shear zones ($v=0.5$; $m=5\times 10^{-4}$).

The following observations can be made, for significant values of the reinforcement coefficient r.

• The results of the comparison between curves n° 1 and n° 2 confirm what could be foreseen from the theory (and from common sense): using the same kind of mechanisms (implying only velocity discontinuities) and taking the shear and bending strengths of the nails into account leads to an increase of the upperbound estimate obtained for the stability factor.

• Comparing curves n° 2 and n° 3 is more interesting: it shows how the evaluation of the stability factor is significantly improved by the consideration of "failure mechanisms" with shear zones once the shear and bending strengths of the nails are taken into account. Furthermore it appears that after the drop induced by these failure mechanisms the estimate obtained for the stability factor remains but slightly superior to that obtained when the reinforcing nails are considered perfectly flexible ($v=0$, $m=0$: curve n° 1).

4.3. Comments

Looking forward to the practical implications of such an analysis it may be convenient to present the results of this study in the same way as in [Blondeau *et al.*, 1984] and

[Jewell and Pedley, 1990], that is, by determining in the plane (N, V) a domain of the allowable forces in the nails ("multicriteria analysis").

It can be noticed that the maximum resisting work developed by a nail in a "double hinge" mechanism, given by formula (32), is equal to that which could be computed in the corresponding mechanism with a velocity discontinuity, given by formula (26), provided that a modified strength condition $f^*(N, V) \leqq 0$ be adopted for the nail. Such a condition is simply derived from the identification of formulas (26) and (32), expressed for any orientation θ of the velocity discontinuity with respect to the normal to the nail, namely

$$U \underset{(N, V)}{\mathrm{Max}} \{ N \sin\theta + V \cos\theta ; f^*(N, V) \leqq 0 \} = U N_0 (|\sin\theta| + \mu |\cos\theta|).$$

This relation shows that the curve of equation $f^*(N, V) = 0$ in the plane (N, V) may be constructed as the convex envelope of a family of straight lines, whose parametric equation writes

$$N \sin\theta + V \cos\theta = N_0 (|\sin\theta| + \mu |\cos\theta|)$$

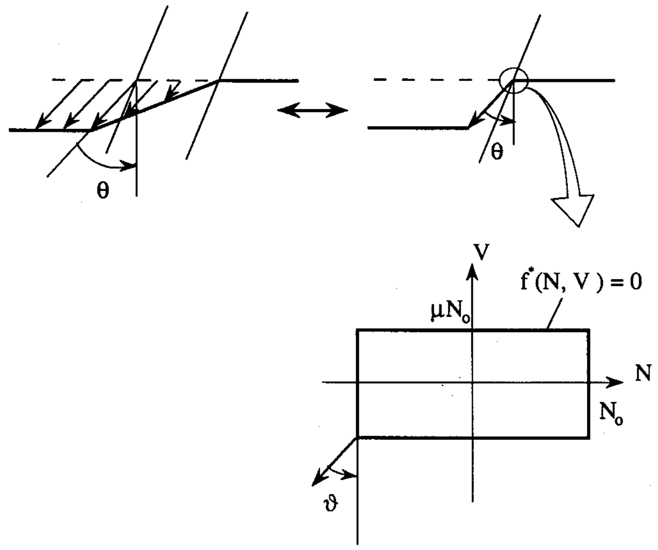

Fig. 15. – Construction of the modified strength condition $f^*(N, V) \leqq 0$.

thus defining the following rectangular domain (*Fig.* 15)

$$f^*(N, V) \leqq 0 \quad \Leftrightarrow \quad \begin{cases} |N| \leqq N_0 \\ |V| \leqq \mu N_0 \end{cases}$$

It is to be reminded that the dimensionless factor $\mu = 2 M_0 / N_0 \delta$ which is involved in this analysis, compares a characteristic dimension of the cross-section of the nail with the length of nail δ intersected by the shear zone. As an example, for a nail with a circular cross-section of diameter d, one gets:

$$\mu = \frac{4}{3\pi} d/\delta.$$

It follows that, the greater δ with respect to such a transversal dimension of the nail, the smaller the value of μ. A typical value of μ is of the order of 1/10.

5. Concluding remarks

Thanks to the use of rotational failure mechanisms involving deformation zones instead of velocity discontinuity surfaces, the yield design kinematic approach results in a more reliable assessment of the stability of soil nailed structures for which shear and bending resistances of the reinforcements are to be considered. More specifically, relying upon clear mechanical arguments, it provides an unambiguous answer to the question of whether and to what extent the contribution due to the shear strength of nails should be taken into account. As an example, a parametric study performed in the case of a purely cohesive soil has shown that stability calculations based upon the use of velocity discontinuity surfaces, which integrate the full amount of the shear strength of nails (curve n° 2 of Figure 14), may significantly overestimate the stability of the structure, since they implicitly disregard any failure mode of the nails by flexion.

On the other hand, such an analysis outlines the possibility of setting up a simplified stability analysis method, in which failure mechanisms with discontinuity surfaces could still be used, *provided that sharply reduced shear strength capacities* (μN_0 instead of V_0) *be taken into account* in the computation of the maximum resisting work for the nails. The initial strength condition of the nail shall then be altered to:

$$f(N, V, M) \leqq 0 \quad \text{along with } |V| \leqq \mu N_0.$$

Anyhow, leaving aside the particular case of reinforcements having large cross sectional dimensions (such as in the micropiling technique) which would need a specific investigation, it turns out that the maximum resisting contribution to the stability of structures which could be expected from the shear component developed in the reinforcements remains quite limited, so that in most cases neglecting such a contribution only reduces the estimate of the stability factor very slightly, *i.e.* leads to a slightly conservative estimate.

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