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Lower and Upper Bound Estimates for the Macroscopic Strength Criterion of Fiber Composite Materials

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Abstract : The formulation of a homogenization procedure within the framework of the yield design theory makes it possible to derive a strength criterion for a fiber composite material in a rigorous way, from the only definitions of the strength properties of the constituents (matrix and fibers) and of their geometrical, structural and voluminal arrangement. Making use of both yield design static and kinematic approaches, quite simple analytical lower and upper bound estimates are obtained for a unidirectional fiber composite. A detailed analysis of those estimates is carried through in the specific case when the composite is subjected to a uniaxial solicitation or to plane strain conditions parallel to the fibers direction .

1. *Introduction*

Among the various theoretical investigations being carried out for analysing the inelastic behaviour of fiber composite materials, some are more specifically concerned with the determination of their ultimate strength properties. They are generally based on a "micro-macro" approach which aims at relating the overall behaviour of the composite to the strength properties of its constituents taking into account their structural arrangement (e.g. respective voluminal proportions, geometry of the fibers etc.) (Saurindranath and Mc Laughlin, 1975 ; Sawicki, 1981 ; Dvorak and Bahei-el-Din, 1982). When the fibers are laid out throughout the matrix material following a regular pattern, a homogenization method formulated within the yield design theory (de Buhan and Salençon, 1987) provides a rigorous mechanical framework for solving such a problem.

This paper is devoted to the application of that method to a unidirectional fiber composite material. The general static and kinematic definitions of the corresponding macroscopic strength criterion are given, and lower and upper bounds are derived from the implementation of the yield design static and kinematic approaches respectively. These bounds can be expressed by means of explicit analytical formulae in case of simple solicitations such as uniaxial or plane strain loading conditions. Their major interest lies in providing easily usable estimates for the overall strength of the composite, which remain valid for any voluminal proportion of the fibers.

2. *General definitions of the strength criterion*

2.1. *Working hypotheses and notations*

Consider a composite material made of one single array of long parallel fibers (unidirectional composite) periodically distributed throughout a homogeneous matrix material in such a way that a rectangular parallepiped of dimensions a_1 , a_2 , a_3 along three orthogonal directions of a $Ox_1x_2x_3$ coordinate system may be exhibited as a representative volume (Fig. 1), called a "unit cell" and denoted by $\mathcal{a}$. This unit cell consists of a cylindrical, not necessarily circular, volume $\mathcal{a}_f$, of fiber material embedded in the matrix material which occupies the remaining part $\mathcal{a}_m$ of the cell.

The constituents are supposed to obey von Mises strength conditions, namely :

$$F_m(\underline{\underline{\sigma}}) = [\, J_2(\underline{\underline{\sigma}})\,]^{1/2} - K_m/\sqrt{3} < 0 \qquad (1)$$

for the matrix, and

$$F_f(\underline{\underline{\sigma}}) = [\, J_2(\underline{\underline{\sigma}})\,]^{1/2} - K_f/\sqrt{3} < 0 \qquad (2)$$

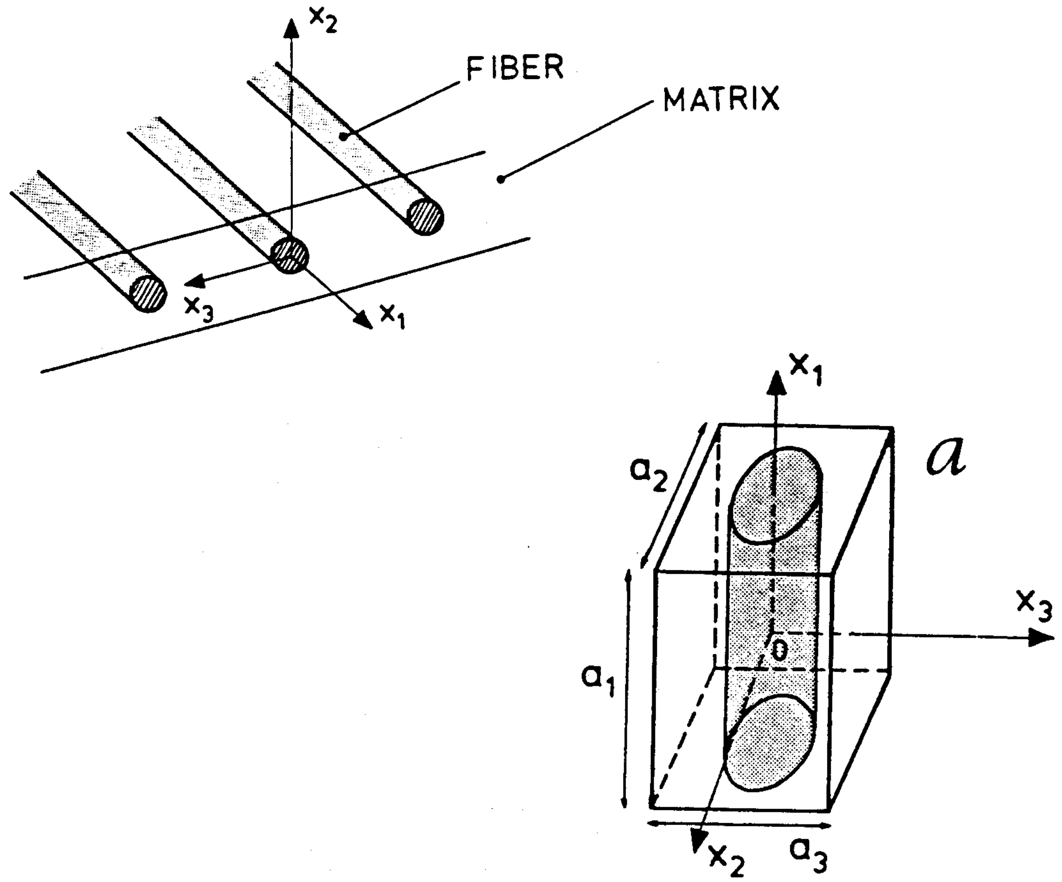

Fig. 1 : Representative unit cell for a unidirectional fiber composite.

for the fiber material, where $J_2(\underline{\underline{\sigma}})$ denotes the second invariant of the deviatoric stress $(J_2(\underline{\underline{\sigma}}) = 1/2 \quad \underline{\underline{s}} : \underline{\underline{s}}^{(\star)})$, with K_m (resp. K_f) representing the uniaxial strength of the matrix (resp. fiber) material.

Furthermore, failure by lack of adherence between the fiber and the surrounding matrix will be discarded in the present analysis, thereby assuming perfect bonding between the constituents.

(⋆) $1/2\ \underline{\underline{s}} : \underline{\underline{s}} = 1/2\ s_{ij}\ s_{ji}$, *with summation over the repeated subscripts.*

2.2. Static definition

According to the yield design homogenization theory (Suquet, 1985) the macroscopic strength condition of the above described composite, considered as a homogeneous material, writes :

$$F_{hom}(\underline{\underline{\Sigma}}) \leq 0 \tag{3}$$

where the macroscopic stress $\underline{\underline{\Sigma}}$ is computed as the volume average over $\mathcal{Q}$ of a stress field $\underline{\underline{\sigma}}$ which must comply with the following conditions :

a) *boundary periodicity*, i.e. the origin O being taken at the center of the parallelepiped

$$\sigma_{ij}\left(x_j = -\frac{a_j}{2}\right) = \sigma_{ij}\left(x_j = +\frac{a_j}{2}\right) \tag{4-a}$$

whatever i and j ;

b) *equilibrium* :

div $\underline{\underline{\sigma}} = 0$ (and $[\![\underline{\underline{\sigma}}]\!] \cdot \underline{n}$ across possible stress discontinuity surfaces) ; (4-b)

c) *strength requirements* :

$$F_m(\underline{\underline{\sigma}}(\underline{x})) \leq 0 \quad \text{(resp. } F_f(\underline{\underline{\sigma}}(\underline{x})) \leq 0) \tag{4-c}$$

whatever $\underline{x}$ belonging to $\mathcal{Q}_m$ (resp. $\mathcal{Q}_f$).

2.3. Kinematic definition

Introducing the support function π_{hom} of the macroscopic strength criterion, defined as (Salençon, 1983) :

$$\pi_{hom}(\underline{\underline{D}}) = \underset{\underline{\underline{\Sigma}}}{\text{Sup}} \{\underline{\underline{\Sigma}} : \underline{\underline{D}} \ ; \ F_{hom}(\underline{\underline{\Sigma}}) \leq 0\} \tag{5}$$

where $\underline{\underline{D}}$ represents any symmetric second order tensor, it can be proved (Suquet, 1985 ; de Buhan, 1986) that :

$$\pi^{hom}(\underline{\underline{D}}) = \operatorname*{Inf}_{\underline{v}} \{<\pi(\underline{\underline{d}})> \; ; <\underline{\underline{d}}> = \underline{\underline{D}}\} \qquad (6)$$

where :

• $\underline{v}$ denotes any velocity field giving a strain rate field $\underline{\underline{d}}$ periodic over $\mathcal{a}$

$$\bullet <\underline{\underline{d}}> = \frac{1}{2\, a_1\, a_2\, a_3} \int_{\partial \mathcal{a}} (\underline{n} \otimes \underline{v} + \underline{v} \otimes \underline{n})\, dS$$

defines the mean value of the strain rate field $\underline{\underline{d}}$ computed over the cell

• and $<\pi(\underline{\underline{d}})>$ is defined by

$$<\pi(\underline{\underline{d}})> = \frac{1}{a_1\, a_2\, a_3} \left(\int_{\mathcal{a}_m} \pi_m(\underline{\underline{d}})\, d\mathcal{a} + \int_{\mathcal{a}_f} \pi_f(\underline{\underline{d}})\, d\mathcal{a} + \int_{S_m} \pi_m(\underline{n} \; ; [\![\underline{v}]\!])\, dS + \int_{S_f} \pi_f(\underline{n} \; ; [\![\underline{v}]\!])\, dS \right)$$

where

$\pi_m(\underline{\underline{d}}) = \operatorname*{Sup}_{\underline{\underline{\sigma}}} \{\underline{\underline{\sigma}} : \underline{\underline{d}} \quad , \quad F_m(\underline{\underline{\sigma}}) \leqslant 0\}$,

$\pi_f(\underline{\underline{d}})$ has the same definition with F_f ,

S denotes any jump surface for $\underline{v}$,

S_m is the portion of S lying in $(\mathcal{a}_m)$,

S_f has the same definition with $(\mathcal{a}_f)$,

$[\![\underline{v}]\!]$ is the jump of $\underline{v}$ when crossing S following its normal $\underline{n}$,

$\pi_m(\underline{n} \; ; [\![\underline{v}]\!]) = \operatorname*{Sup}_{\underline{\underline{\sigma}}} \{\underline{n} \cdot \underline{\underline{\sigma}} \cdot [\![\underline{v}]\!] \quad , \quad F_m(\underline{\underline{\sigma}}) \leqslant 0\}$,

$\pi_f(\underline{n} \,;\, [\![\underline{v}]\!])$ has the same definition with F_f .

Generally speaking, the determination of the strength condition (3) would require the use of heavy numerical methods implementing either of the two above definitions. Fortunately the specific geometry of the unit cell in the case of a unidirectional composite will be exploited to derive relevant, though simple, analytical estimates for the actual macroscopic condition.

3. *Lower bound estimate*

Referring to the static definition given by Eq. (3) and (4-*a*) to (4-c), it is possible to exhibit piecewise constant stress fields satisfying all the above prescribed conditions :

$$\underline{\underline{\sigma}}(\underline{x}) = \begin{cases} \underline{\underline{\sigma}}^m & \forall \underline{x} \in \mathcal{A}_m \\ \underline{\underline{\sigma}}^m + \sigma^f \, \underline{e}_1 \otimes \underline{e}_1 & \forall \underline{x} \in \mathcal{A}_f \end{cases} \qquad (7)$$

with

$$F_m(\underline{\underline{\sigma}}^m) \leq 0 \quad \text{and} \quad F_f(\underline{\underline{\sigma}}^m + \sigma^f \, \underline{e}_1 \otimes \underline{e}_1) \leq 0 \qquad (8)$$

($\underline{e}_1$: unit vector of Ox_1 , parallel to the fiber).

This particular form of stress fields automatically complies with the periodicity condition (4-*a*) and the equilibrium equation as well since it ensures the continuity of the stress vector acting at any point upon the interface between the matrix and the fiber.

As a matter of fact, whatever $\underline{n}$ such that $\underline{n} \cdot \underline{e}_1 = 0$

$$[\![\underline{\underline{\sigma}}]\!] \cdot \underline{n} = (\sigma^f \, \underline{e}_1 \otimes \underline{e}_1) \cdot \underline{n} = \sigma^f \, \underline{e}_1 (\underline{e}_1 \cdot \underline{n}) = 0 \, .$$

Denoting by η the voluminal percentage of the fibers the macroscopic stress $\underline{\underline{\Sigma}}$ defined as the volume average of such a piecewise constant stress field writes :

$$\underline{\underline{\Sigma}} = \underline{\underline{\sigma}}^m + \eta\, \sigma^f\, \underline{e}_1 \otimes \underline{e}_1 \;. \tag{9}$$

It can be easily shown (see Appendix) that *sufficient* conditions for $\underline{\underline{\Sigma}}$ to satisfy the macroscopic criterion (3) are :

$$F_m(\underline{\underline{\sigma}}^m) \leq 0 \quad \text{and} \quad |\sigma^f| \leq K_f - K_m \;. \tag{10}$$

It follows immediately that the condition :

$$F^-(\underline{\underline{\Sigma}}) \leq 0$$

with :

$$F^-(\underline{\underline{\Sigma}}) = \operatorname*{Inf}_{\sigma^f} \{F_m(\underline{\underline{\Sigma}} - \eta\, \sigma^f\, \underline{e}_1 \otimes \underline{e}_1)\ ;\ |\sigma^f| \leq K_f - K_m\} \tag{11}$$

constitutes a safe lower bound approximation to the actual macroscopic condition :

$$F^-(\underline{\underline{\Sigma}}) \leq 0 \quad \Rightarrow \quad F_{hom}(\underline{\underline{\Sigma}}) \leq 0 \;.$$

4. *Upper bound estimates*

4.1. A first upper bound estimate can immediately be deduced from the kinematic definition (6). Considering the homogeneous strain rate field $\underline{\underline{d}}(\underline{x}) = \underline{\underline{D}} \quad \forall \underline{x} \in \mathcal{Q}$ derived from a velocity field in the form

$$\underline{v} = \underline{\underline{D}} \cdot \underline{x}$$

one gets :

$$<\pi(\underline{\underline{d}})> = (1 - \eta)\ \pi_m(\underline{\underline{D}}) + \eta\ \pi_f(\underline{\underline{D}})$$

with

$$\pi_m(\underline{\underline{D}}) = \begin{cases} K_m \left(\frac{2}{3}\, \underline{\underline{D}} : \underline{\underline{D}}\right)^{1/2} & \text{if} \quad \operatorname{tr} \underline{\underline{D}} = 0 \\ +\infty & \text{otherwise} \end{cases}$$

(analogous formulae for $\pi_f(\underline{\underline{D}})$).

Hence for any $\underline{\underline{D}}$

$$\pi_{hom}(\underline{\underline{D}}) \leq [(1-\eta)\, K_m + \eta\, K_f]\ (2/3\ \underline{\underline{D}} : \underline{\underline{D}})^{1/2}\,.$$

This proves that the strength capacities of the homogenized composite are inferior to those that would be exhibited by an isotropic von Mises material obtained by averaging the strength characteristics of the constituents.

In other words :

$$F_{hom}(\underline{\underline{\Sigma}}) \leq 0 \quad \Rightarrow \quad F_1^{+}(\underline{\underline{\Sigma}}) \leq 0 \qquad (12\text{-}a)$$

with

$$F_1^{+}(\underline{\underline{\Sigma}}) = [J_2(\underline{\underline{\Sigma}})]^{1/2} - \langle K\rangle/\sqrt{3} \qquad (12\text{-}b)$$

$$\langle K\rangle = (1-\eta)\, K_m + \eta\, K_f \quad .$$

4.2. Other upper bound estimates can be obtained by considering piecewise constant velocity fields in which a discontinuity plane parallel to the fibers and running across the matrix divides the unit cell into two blocks. Due to the periodicity condition, such a plane is prescribed to be perpendicular to the Ox_i axis, $i = 2$ or 3 (Fig. 2). Denoting by $\underline{V}$ the velocity jump across this plane, one gets :

$$\underline{\underline{D}} = \langle\underline{\underline{d}}\rangle = \frac{1}{2a_i}\,(\underline{e}_i \otimes \underline{V} + \underline{V} \otimes \underline{e}_i) \qquad i = 2\,,3$$

and

$$\langle\pi(\underline{\underline{d}})\rangle = \frac{1}{a_i}\,\pi_m(\underline{e}_i\ ;\ \underline{V})$$

with, as the matrix obeys a von Mises criterion :

$$\pi_m(\underline{e}_i\ ;\ \underline{V}) = \begin{cases} K_m\ |\underline{V}|/\sqrt{3} & \text{if} \quad \underline{V}\cdot\underline{e}_i = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

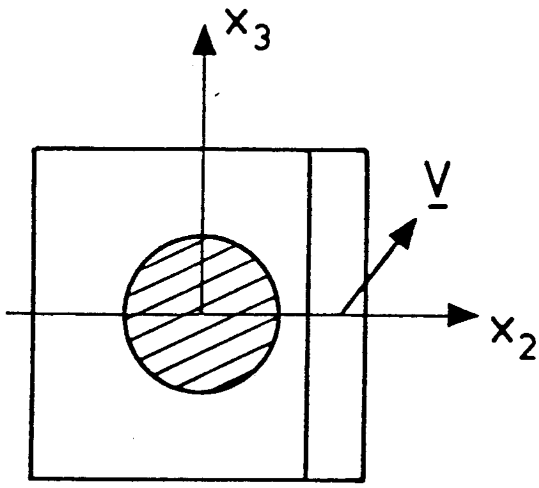

Fig. 2 : Velocity field involving a jump plane passing through the matrix.

The kinematic definition of the macroscopic criterion expressed by Eq. (5) and (6) leads to the following implication

$$F_{hom}(\underline{\underline{\Sigma}}) < 0 \Rightarrow \underline{\underline{\Sigma}} : <\underline{\underline{d}}> \leq <\pi(\underline{\underline{d}})>$$

and therefore

$$F_{hom}(\underline{\underline{\Sigma}}) \leq 0 \Rightarrow F_i^+(\underline{\underline{\Sigma}}) \leq 0 \qquad (13\text{-}a)$$

where for $i = 2, 3$

$$F_i^+(\underline{\underline{\Sigma}}) \leq 0 \Leftrightarrow (\underline{\underline{\Sigma}} \cdot \underline{e}_i) \cdot \underline{V} \leq \frac{K_m}{\sqrt{3}} |\underline{V}| \qquad (13\text{-}b)$$

whatever $\underline{V}$ such that $\underline{V} \cdot \underline{e}_i = 0$ (*) .

(*) $F_i^+(\underline{\underline{\Sigma}})$ *may be expressed by*

$$F_i^+(\underline{\underline{\Sigma}}) = \operatorname*{Sup}_{\underline{V}} \{(\underline{\underline{\Sigma}} \cdot \underline{e}_i) \cdot \underline{V} - \frac{K_m}{\sqrt{3}} |\underline{V}| \; ; \; \underline{V} \cdot \underline{e}_i = 0\} .$$

5. Application to simple loading conditions

5.1. "Plane strain" criteria

The concept of a "strength criterion under plane strain conditions" within the strict framework of the yield design theory, i.e. without any explicit reference to a constitutive law, has been introduced and throroughly discussed in (Salençon, 1983). It can be briefly exposed as follows in the present case.

The support function π_{hom} defined by Eq. (5) makes it possible to determine the macroscopic strength criterion through the dual formulation :

$$F_{hom}(\underline{\underline{\Sigma}}) \leq 0 \iff \sup_{\underline{\underline{D}}} \{\underline{\underline{\Sigma}} : \underline{\underline{D}} - \pi_{hom}(\underline{\underline{D}})\} \leq 0 \qquad (14)$$

(cf. Frémond and Friaâ, 1978).

If the latter formula is used with the considered $\underline{\underline{D}}$ being restricted to plane strain rate tensors parallel to Ox_1x_2 , i.e. satisfying the conditions :

$$D_{3i} = D_{i3} = 0 \quad , \quad i = 1 , 2 , 3 \qquad (15)$$

it comes out that explicitating Eq. (16)

$$\sup_{\underline{\underline{D}}} \{\underline{\underline{\Sigma}} : \underline{\underline{D}} - \pi_{hom}(\underline{\underline{D}}) \; ; \; \underline{\underline{D}} \text{ satisf. Eq. (15)}\} \leq 0 \qquad (16)$$

will provide an upper bound estimate for $F_{hom}(\underline{\underline{\Sigma}})$ where only the components Σ_{11} , $\Sigma_{12} = \Sigma_{21}$, and Σ_{22} of $\underline{\underline{\Sigma}}$ are concerned.

As a matter of fact, Eq. (16) gives the exact restrictions imposed to those components when generated by any $\underline{\underline{\Sigma}}$ under the condition $F_{hom}(\underline{\underline{\Sigma}}) \leq 0$. Eq. (16) can also be seen as the projection of the strength domain defined by Eq. (3) onto the subspace of the Σ_{ij} (i , j = 1 , 2) components of $\underline{\underline{\Sigma}}$.

It can (rather abusively) be named the "plane strain strength criterion" of the composite material.

The purpose of the present section is to derive lower and upper bound estimates for the "plane strain strength criterion" from the results obtained before.

Let then Σ_I and Σ_{II} with $\Sigma_I \geqslant \Sigma_{II}$ be defined as the principal values of the two-dimensional "stress tensor" generated by the components Σ_{11} , $\Sigma_{12} = \Sigma_{21}$, Σ_{22} in plane Ox_1x_2 , and θ the inclination of the major principal axis Σ_I to the fibers direction Ox_1 . After some calculations it turns out that the previously obtained lower and upper bounds write as follows.

- *Lower bound estimate (11)*

$$F^-(\Sigma_I , \Sigma_{II} , \theta) = \Sigma_I - \Sigma_{II} - K^-(\theta) \leqslant 0 \qquad (17)$$

with putting $k = \langle K \rangle / K_m \geqslant 1$

$$K^-(\theta) = K_m \begin{cases} (k-1) \; |\cos 2\theta| + \sqrt{4/3 - [(k-1)^2 \sin^2 2\theta]} \\ \qquad \text{if} \quad |\tan 2\theta| \leqslant 2/[(k-1)\sqrt{3}] \\ 2/(\sqrt{3} \sin 2\theta) \qquad \text{otherwise} . \end{cases}$$

- *Upperbound estimate (12)*

$$F_1^+(\Sigma_I , \Sigma_{II} , \theta) = \Sigma_I - \Sigma_{II} - K_1^+(\theta) \leqslant 0 \qquad (18)$$

where $K_1^+(\theta) = 2kK_m/\sqrt{3}$.

- *Upperbound estimate (13)*

For $i = 2$, $\underline{V} = V \underline{e}_1$ and taking Eq. (14) into account (13-b) becomes

$$\Sigma_{12} V \leqslant \frac{K_m}{\sqrt{3}} |V|$$

that is

$$|\Sigma_{12}| < \frac{K_m}{\sqrt{3}}$$

or

$$F_2^+(\Sigma_I\ ,\ \Sigma_{II}\ ,\ \theta) = \Sigma_I - \Sigma_{II} - K_2^+(\theta) < 0 \qquad (19)$$

where

$$K_2^+(\theta) = 2K_m / (\sqrt{3}\ \sin 2\,\theta)$$

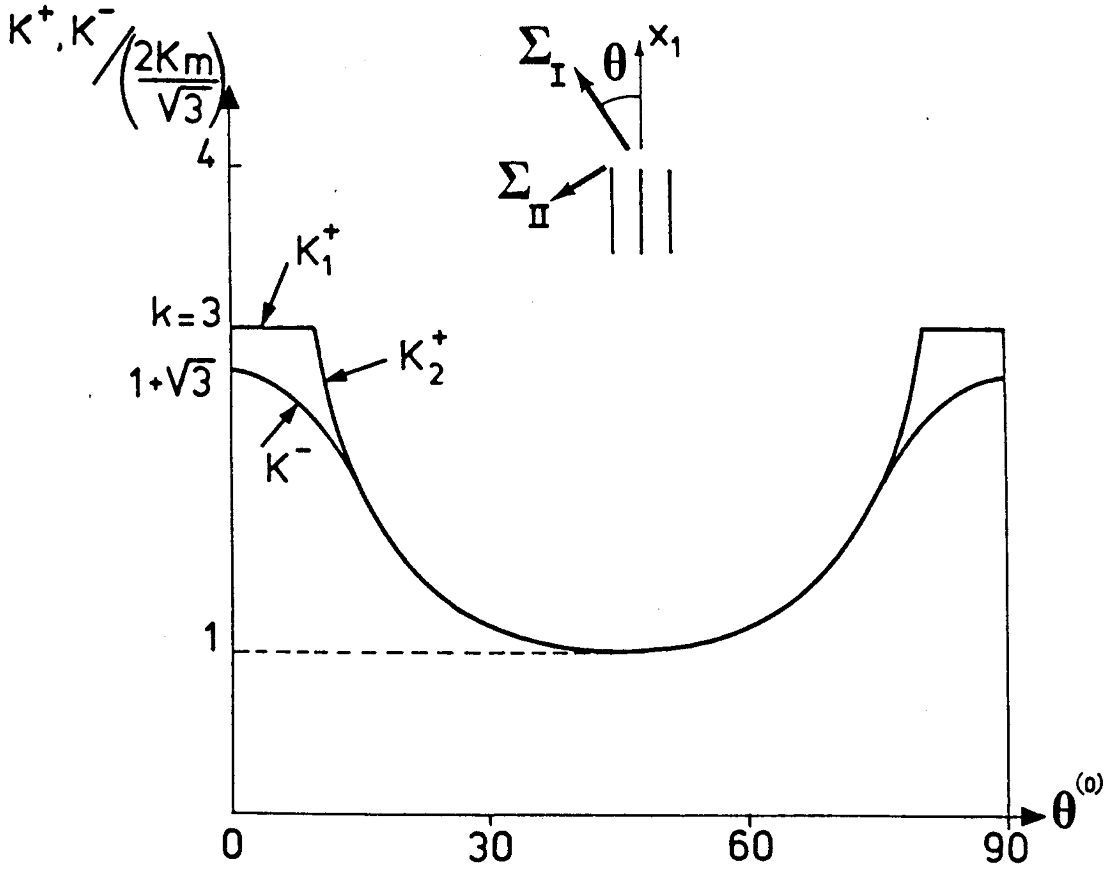

Fig. 3 : Upper and lower bound estimates for the strength of a unidirectional fiber composite under plane strain conditions.

The variations of K^- and $K^+ = \text{Min}\,\{K_1^+ , K_2^+\}$ as functions of the orientation θ are plotted in Fig. 3 where the numerical value of $k = 3$ has been taken by way of illustration. It appears that the relative difference between the two bounds remains lower than 20 % and even decreases to zero for θ ranging between 15° and 75°. It is worth noting in particular that the overall strength of the composite reduces to that of the matrix alone when the principal stresses are inclined at $\theta = 45°$ to the fibers direction.

5.2. Bounds on the uniaxial strength

The macroscopic solicitation writes in this case :

$$\underline{\underline{\Sigma}} = \Sigma\, \underline{e}_\theta \otimes \underline{e}_\theta$$

with $\underline{e}_\theta = \cos\theta\, \underline{e}_1 + \sin\theta\, \underline{e}_2$.

Denoting by $\Sigma^\star(\theta)$ the uniaxial tensile strength in the direction $\underline{e}_\theta$, defined by :

$$F_{hom}(\pm\, \Sigma^\star(\theta)\, \underline{e}_\theta \otimes \underline{e}_\theta) = 0$$

the strength condition for this particular type of solicitation can be written

$$F_{hom}(\underline{\underline{\Sigma}} = \Sigma\, \underline{e}_\theta \otimes \underline{e}_\theta) \leq 0 \iff |\Sigma| \leq \Sigma^\star(\theta) \ .$$

The following lower and upper bound estimates can be derived from the preceding results.

- *Lower bound*

From Eq. (11) one gets

$$\Sigma^-(\theta) \leq \Sigma^\star(\theta) \qquad (20)$$

where :

$$\Sigma^{-}(\theta) = K_m \begin{cases} (k - 1)\left(1 - \frac{3}{2}\sin^2\theta\right) + \sqrt{1 - 3(k - 1)^2\sin^2\theta\left(1 - \frac{3}{4}\sin^2\theta\right)} \\ \qquad \text{if} \quad 0 \leq \theta \leq \theta^{\star} \\ \frac{1}{\sin\theta} \Big/ \sqrt{3\left(1 - \frac{3}{4}\sin^2\theta\right)} \quad \text{if} \quad \theta^{\star} \leq \theta \leq 90° \end{cases}$$

where $\theta^{\star}$ is the value of θ for which the two above expressions give the same result.

This lower bound estimate decreases continuously from $\Sigma^{-} = k\,K_m$ for $\theta = 0°$ to its minimum value $\Sigma^{-} = K_m$ reached for $\theta \approx 57°,2$ $(\sin\theta = \sqrt{2/3})$, then increases again slightly up to $\Sigma^{-} = 2\,K_m / \sqrt{3}$ when $\theta = 90°$ (transverse loading) : see fig. 4.

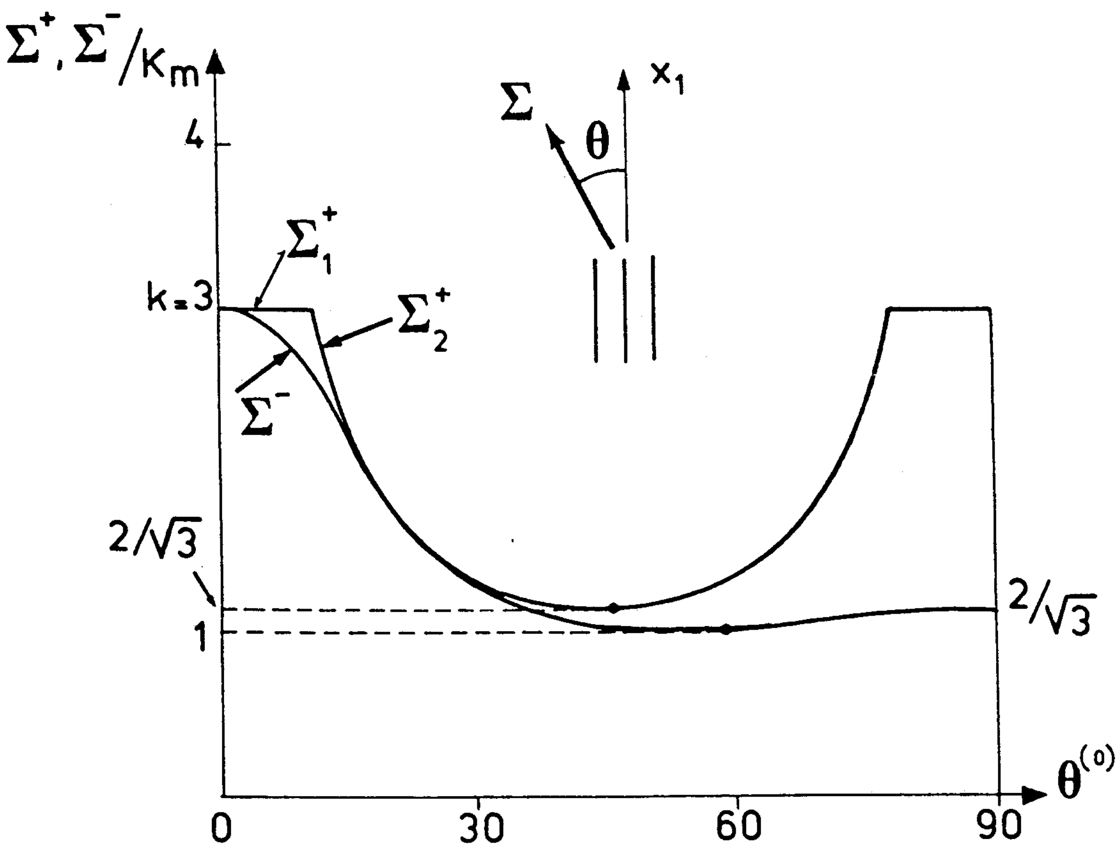

Fig. 4 : Upper and lower bounds for the uniaxial strength of a unidirectional composite.

• *Upper bounds*

A first upper bound estimate of $\Sigma^*(\theta)$ derived from Eq. (12) is simply :

$$\Sigma^*(\theta) \leq \Sigma_1^+ = k\ K_m \tag{21}$$

Making use of Eq. (13) with $i = 2$ and $\underline{V} = V\,\underline{e}_1$ yields immediately :

$$|\Sigma_{12}| \leq \frac{K_m}{\sqrt{3}}$$

and the upper bound $\Sigma_2^+(\theta)$:

$$\Sigma^*(\theta) \leq \Sigma_2^+(\theta) = \frac{2\ K_m}{\sqrt{3}\ \sin 2\theta}\ . \tag{22}$$

The corresponding curves are drawn in Fig. 4 for $k = 3$. Unlike the case of plane strain conditions the gap between the two bounds widens considerably as soon as $\theta > 60°$. The actual value of the transverse strength for instance is bounded by $\Sigma^-(90°) = 2K_m/\sqrt{3}$ and $\Sigma^+(90°) = k\ K_m$.

However it can be proved (de Buhan and Taliercio, 1988 ; Taliercio, 1989) that these bounds turn out to be the exact value of the uniaxial strength of the composite for $\eta \ll 1$ (reinforcement by thin fibers) and $\eta = 1$ (matrix voluminal proportion reduced to zero) respectively. Therefore as a first approach, the uniaxial strength of the composite could be reasonably well approximated by the following interpolation formula :

$$\Sigma^*(\theta) = (1 - \eta)\ \Sigma^-(\theta) + \eta\ \Sigma^+(\theta)\ .$$

The validity of such a heuristic formula should be confirmed by a more elaborate analysis of the problem resorting for instance to numerical methods in order to solve the yield design problem defined over the unit cell and thereby to determine the macroscopic strength condition (Marigo and al., 1987 ; case of periodic porous media).

6. Conclusion

The yield design homogenization method has been successfully applied to the determination of upper and lower bounds for the macroscopic strength of a unidirectional fiber composite material. These bounds may be expressed by means of explicit analytical formulae in two circumstances : plane strain loading of the composite in a direction parallel to the fibers on the one hand and uniaxial solicitation on the other hand. The two bounds remain close to each other in the first case, thus providing a good estimate for the actual strength of the composite, whereas they diverge considerably in the second case when the composite is subjected to transverse loading. Such a drawback could be partially overcome by proposing a semi-empirical formula which linearly interpolates between the two bounds.

Appendix

Let $\underline{\underline{\Sigma}}$ be a macroscopic stress such that

$$\underline{\underline{\Sigma}} = \underline{\underline{\sigma}}^m + \eta\, \sigma^f\, \underline{e}_1 \otimes \underline{e}_1 \tag{9}$$

with

$$F_m(\underline{\underline{\sigma}}^m) = [J_2(\underline{\underline{\sigma}}^m)]^{1/2} - K_m/\sqrt{3} \leq 0 \tag{10-a}$$

and

$$|\sigma^f| \leq K_f - K_m \quad . \tag{10-b}$$

It comes out immediately :

$$F_f(\underline{\underline{\sigma}}^m + \sigma^f\, \underline{e}_1 \otimes \underline{e}_1)$$

$$= [J_2(\underline{\underline{\sigma}}^m + \sigma^f\, \underline{e}_1 \otimes \underline{e}_1)]^{1/2} - K_f/\sqrt{3}$$

$$\leq [J_2(\underline{\underline{\sigma}}^m)]^{1/2} + [J_2(\sigma^f\, \underline{e}_1 \otimes \underline{e}_1)]^{1/2} - K_f/\sqrt{3}$$

since $[J_2(\underline{\underline{\sigma}})]^{1/2}$ can be regarded as a norm for the deviatoric stress $\underline{\underline{s}}$.

Consequently on account of (10) :

$$F_f(\underline{\underline{\sigma}}^m + \sigma^f \underline{e}_1 \otimes \underline{e}_1) \leq \frac{|\sigma^f|}{\sqrt{3}} - \frac{(K_f - K_m)}{\sqrt{3}} \leq 0 \quad .$$

This last inequation shows that conditions (9) and (10) are sufficient to exhibit a piecewise constant stress field such as (7) in equilibrium with $\underline{\underline{\Sigma}}$ and satisfying the strength conditions defined by (8).

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