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An introduction to the yield design theory and its applications to soil mechanics

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ABSTRACT. – The design of new types of earth structures using recent industrial techniques as soil reinforcement, soil nailing, etc., makes it desirable to overcome some mechanical inconsistencies which may still be encountered in current stability analysis methods. The paper is devoted to a presentation of the yield design approaches starting from the obvious necessary stability condition that the equilibrium of the considered structure and the resistance of the constitutive soil should be compatible. The static approach of the yield design theory follows directly from this condition, leading to lower estimates of the extreme loads. It is shown that the corresponding kinematic approach must be derived by dualizing the static approach through the principle of virtual work so that mechanical consistency is ensured: the concept of maximum resisting work is introduced and the method can be efficiently used to obtain upper bound estimates for the extreme loads. Most of the arguments of the paper are developed for a classical stability analysis problem, making it possible to illustrate the theoretical concepts and to thoroughly examine the practical implementation of the methods. Various fields of applications are finally surveyed.

1. The concepts of failure and yield strength in soil mechanics

The appearance of soil mechanics as an engineering science is usually associated with Coulomb's 1773 celebrated memoir. Devoted both to structures and to soil mechanics it dealt with the stability of pillars and vaults as well as with the calculation of earth pressure on a retaining wall. It introduced yield design methods, which might have been first perceived in Galileo's 1638 study of the cantilever beam, and were to dominate most soil mechanics analyses for the next 150 years. These methods are still efficiently used for stability analyses of slopes, tunnels, etc., and lead to new developments when reinforced soil structures are concerned.

The yield design approach is based upon the following fundamental concepts:

- soil is modelled as a classical three dimensional *continuum* notwithstanding it being both particulate (e.g. clays) or granular (e.g. sands) and polyphasic (particles, water, air);
- this material (the modelled soil) is characterized by *a strength-criterion* which defines local failure or yielding of the continuum;
- the analysis is performed with the purpose of investigating if equilibrium under the applied loads – dead or active – can comply with the limitation of the strength-criterion being imposed at every point of the structure under consideration.

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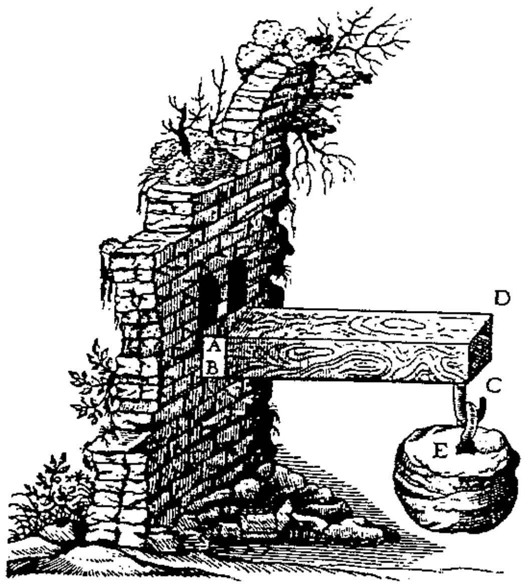

Fig. 1. – The cantilever beam studied by Galileo [1638]

Yield strength can therefore be considered the first concept introduced by civil engineers for the constitutive model of soil as a material: an incomplete description limiting the internal force intensity from the risk of failure.

The concept itself together with the thought process and reasonings it involves certainly came out of observing failure patterns encountered in civil engineering. Looking at landslides or earth-fill ruptures, or at some classical laboratory experiments often provides evidence of the localization of deformation in soil layers whose thickness is very small when compared with a typical length of the studied structure or specimen. The failure mechanism may then be interpreted with these "shear bands", as they are often called [Desrues, 1984], being regarded as "slip-surfaces" and the rest of the body under consideration staying motionless or being given a rigid body motion. Thence the concept of mobilized strength at any point of a failure surface, stability requiring that it should at least balance the driving forces.

Those are indeed the very foundations of yield design methods. Unfortunately when applying them in common practice, civil engineers departing from Coulomb's sound reasoning have fairly often introduced some inconsistencies from the mechanics point of view [Coussy & Salençon, 1979] which don't seem to have been of consequence – except for *glasnost's* sake – when classical circumstances were encountered.

There might be objections about the relevance of such analyses at this time based upon a bare outline of the constitutive laws of soils while sophisticated constitutive models are being developed and numerical codes are available. As a matter of fact both types of approaches are not to be substituted for each other. They will continue to provide project engineers with significant data from differing points of view: for instance one may observe that once rupture of a structure has occurred, backwards calculations are systematically undertaken within the framework of yield design methods. Nevertheless as new types of earth structures are being designed, applying recent techniques such as reinforced soils, it appears that the mechanical deficiencies referred to previously can no longer be overlooked.

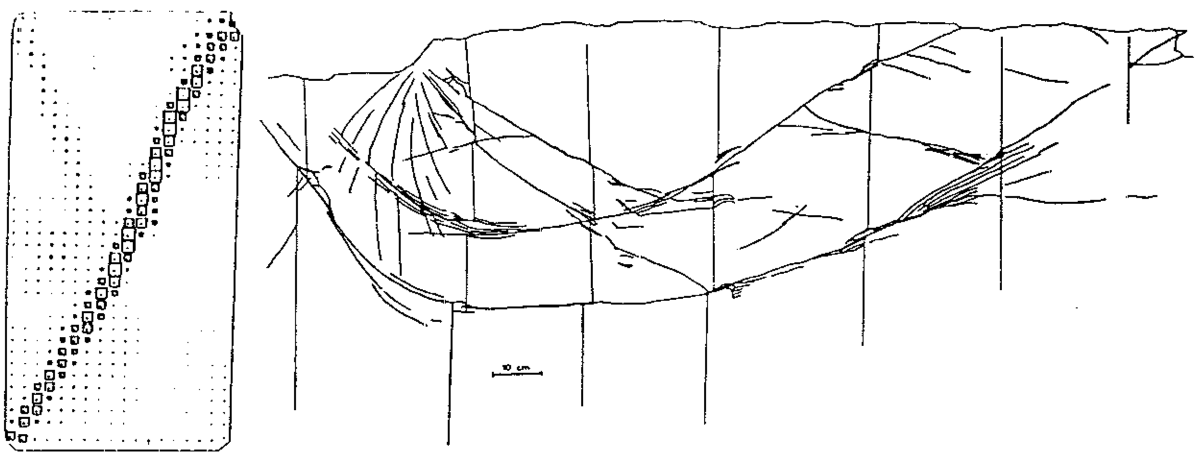

Fig. 2 — Localization of deformation at failure: plane strain punch indentation problem [Habib, 1984], distorsion map for a plane strain compression test [Desrues, 1984].

The goal of this paper is to present a sound basis for the theory of yield design, as fundamental work in convex analysis initiated by Moreau [1966] has made this possible, keeping the mathematical concepts introduced to a minimum while leading to consistent methods for stability analyses.

2. An example of the yield design approach

2.1. Stability analysis of a vertical cut

Before getting to the general formulation of the theory of yield design an example will be considered to illustrate its purposes.

The stability of the homogeneous vertical cut represented in Figure 3 is to be examined under its own weight with the soil obeying Coulomb's isotropic strength condition with cohesion C and friction angle φ.

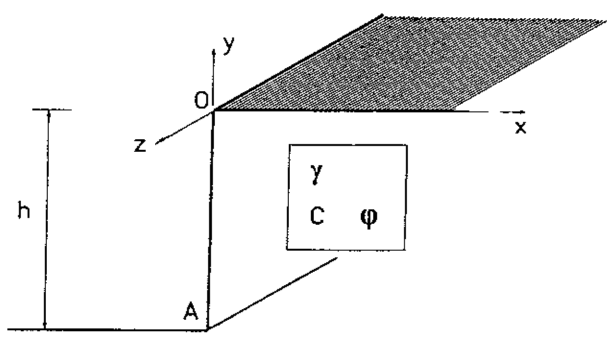

Fig. 3 — Stability of a homogeneous vertical cut under its own weight.

Following Coulomb's original reasoning which is apparent from the very title of the 1773 memoir the analysis will be performed within the framework of statics.

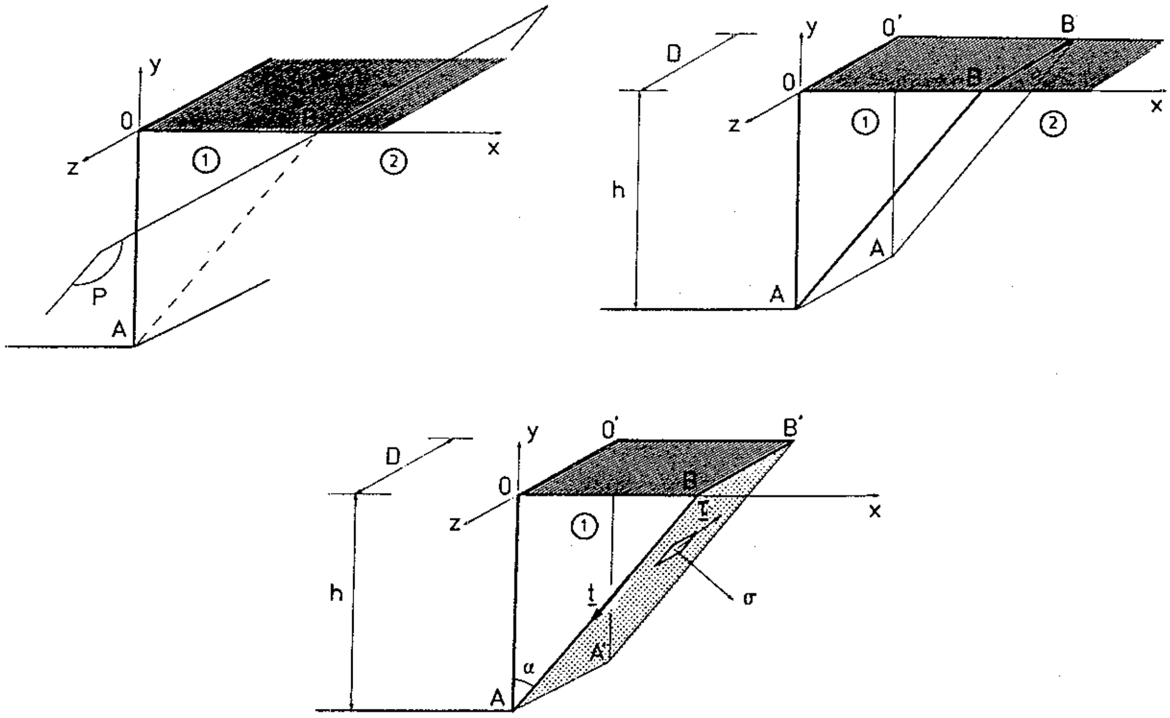

Fig. 4. — Global equilibrium analysis of volume (1).

Let the bulk of soil be virtually separated into two parts (1) and (2) by means of an arbitrary plane (P) passing through the bottom line of the cut as drawn in Figure 4. So that failure of the cut doesn't occur it is necessary that, for any such plane (P), the global equilibrium — *i. e.* from the statics of rigid bodies point of view — of volume (1) or volume (2) be possible under the action of the vertical forces due to gravity (bulk weight of the soil) and of the resisting forces developed by the material.

The cut being considered of infinite length along Oz, the resisting forces developed on the sections parallel to OAB at both ends at infinity are negligible compared with those along the section by plane (P): the analysis can therefore be restricted to checking the stability of a slice with a thickness D along Oz as sketched in Figure 4 assuming no resisting forces on OAB and O′A′B′.

Considering the equilibrium of volume (1) submitted to gravity forces acting throughout the volume and to normal and tangential stresses acting on the surface ABB′A′, the following equations shall be satisfied:

$$\frac{\gamma h^2 \mathrm{D}}{2} \tan\alpha \sin\alpha = -\int_{\mathrm{ABB'A'}} \sigma \, dS \tag{2.1}$$

$$\frac{\gamma h^2 \mathrm{D}}{2} \sin\alpha \, \mathbf{t} = -\int_{\mathrm{ABB'A'}} \tau \, dS \tag{2.2}$$

where σ and τ (a vector) are the normal and tangential components of the stress vector acting at each point of ABB′A′, tensile stresses are counted positive, and $\mathbf{t}$ is the unit vector along **BA**.

Under Coulomb's strength condition, whatever the point in ABB′A′, σ and τ must comply with the inequality

$$(2.3) \qquad |\tau| \leqq C - \sigma \tan \varphi.$$

It follows from Eq. (2.2) and (2.3) that

$$(2.4) \qquad \frac{\gamma h^2 D}{2} \sin \alpha \leqq \int_{ABB'A'} |\tau|\, dS \leqq \int_{ABB'A'} (C - \sigma \tan \varphi)\, dS$$

then through Eq. (2.1)

$$(2.5) \qquad \gamma h/C \leqq 2\,\cancel{\sin}\,\cos \varphi/\sin \alpha \cos(\alpha + \varphi),$$

establishing a necessary condition for the stability of the cut, to be satisfied whatever the plane (P).

The minimum of the second hand of Ineq. (2.5) with respect to α which provides the strongest necessary condition on $\gamma h/C$, is obtained for $\alpha = \pi/4 - \varphi/2$ which yields the inequality

$$(2.6) \qquad \gamma h/C \leqq 4 \tan(\pi/4 + \varphi/2),$$

wyth the conclusion that *any cut with a characteristic ratio* ($\gamma h/C$) *greater than* $4\tan(\pi/4 + \varphi/2)$ *will be unstable.*

2.2. Comments

This example deserves some comments.

- It may be noticed that global equilibrium of volume (1) has only been written partially, that is with respect to the resulting force of the vertical weight $(\gamma h^2 D \tan \alpha)/2$ and of the stresses on ABB′A′; to express it thoroughly, attention should also be paid to the resulting moment, but this is not required here since only a *necessary* condition for stability is established.

- The reasoning in the analysis does not completely make use of the basic idea which led to that necessary condition, namely: the equilibrium for the system under consideration and the material yield-strength should be mathematically compatible. Instead of restricting the study to a partition of the system into two volumes and checking the compatibility between the material yield-strength and *global* equilibrium of one of them, the method could be more thoroughly applied within the framework of continuum mechanics: Coulomb's strength-condition would be written in the form of a criterion on the stress tensor $\underline{\underline{\sigma}}(\mathbf{x})$ at current point $\mathbf{x}$ in the system

$$(2.7) \qquad f(\underline{\underline{\sigma}}(\mathbf{x})) = \sup_{i\, j=1,2,3} \{\sigma_i(1 + \sin\varphi) - \sigma_j(1 - \sin\varphi) - 2C\cos\varphi\} \leqq 0$$

where σ_i denote the principal values of $\underline{\underline{\sigma}}(\mathbf{x})$ and which is equivalent to Ineq. (2.3) written at point $\mathbf{x}$ for any orientation of the considered plane, while equilibrium would refer to the equations defining statically admissible stress fields:

the indefinite equilibrium equations

(2.8) $$\operatorname{div} \underline{\underline{\sigma}} + \rho \mathbf{F} = 0$$

(where $\rho \mathbf{F}$ stands for the volumetric forces) and the boundary conditions on the stresses

• Bringing together the schema of Figure 4 used for the static approach just performed and the often observed failure mechanisms as shown in Figure 2 where evidence of localization leads one to imagine that rigid blocks are slipping with respect to one another, suggests the introduction of an intuitive kinematic reasoning: taking the partition of Figure 4 again it is assumed that failure of the cut will occur it the work of the external forces due to gravity exceeds the work of the resisting forces mobilized along the plane (P) in a rigid body sliding motion along (P). As a matter of fact it can be proved that such an approach will not be acceptable as it is not mechanically consistent with the original static approach (unless $\varphi = 0$) which means that contradictions may arise between results obtained eitherway. Nevertheless the idea of a kinematic approach shall be retained but, as it will appear in the following section, it will proceed from the static approach through the principle of virtual work.

3. Fundamentals of the yield design theory

3.1. The studied problem

The presentation will be restricted to the main results of the theory of yield design for applications to soil mechanics problems; a more detailed statement may be found in [Salençon, 1983].

The geometry of the system under consideration is given, Ω and S being respectively the volume and the boundary. Notations are as follows:

$\underline{\underline{\sigma}}$ denotes a stress field and $\underline{\underline{\sigma}}(\mathbf{x})$ its local value at point $\mathbf{x}$,

$\mathbf{U}$ a velocity field and $\mathbf{U}(\mathbf{x})$ its local value,

$\underline{\underline{\mathbf{d}}}$ the strain rate field associated with $\mathbf{U}$,

$[\![\mathbf{U}(\mathbf{x})]\!]$ denotes the jump of field $\mathbf{U}$ when crossing a velocity discontinuity surface Σ at point $\mathbf{x}$ following the normal $\mathbf{n}(\mathbf{x})$.

The system is loaded according to a loading process depending linearly on n scalar parameters Q_j, the components of the load vector $\mathbf{Q}$. The kinematic vector $\dot{\mathbf{q}}$ is associated with $\mathbf{Q}$ when expressing the work of the external forces in a kinematically admissible velocity field $\mathbf{U}$.

The behaviour of the constitutive material is only known through the strength characteristics: at any point $\mathbf{x}$ of Ω a domain $G(\mathbf{x})$ is given in the stress-space ($\mathscr{R}^6$) defining the allowable stress-tensors As a rule $\underline{\underline{\sigma}}(\mathbf{x}) = 0$ is allowable, and $G(\mathbf{x})$ is star-shaped with respect to 0 and most often a convex set. The domain $G(\mathbf{x})$ is usually defined by means of a scalar strength criterion:

(3.1) $$f(\mathbf{x}; \underline{\underline{\sigma}}(\mathbf{x})) \leq 0 \quad \Leftrightarrow \quad \underline{\underline{\sigma}}(\mathbf{x}) \text{ allowable at point } \mathbf{x}.$$

An example of such a criterion has been given in Section 2 through Eq. (2.7) expressing Coulomb's strength criterion.

The preceding data define the general problem of the yield design theory; the question to be answered is *whether the system, given its geometry and the strength characteristics of the material, will be "stable" under a given load* $\mathbf{Q}$ of $\mathscr{R}^n$.

3.2. Static approach "from inside"

As already seen on the example of Section 2 a necessary condition for the stability of the system is:

$$\text{COMPATIBILITY between} \begin{cases} \text{EQUILIBRIUM under } \mathbf{Q} \\ \text{material STRENGTH characteristics} \end{cases} \tag{3.2}$$

Any load for which condition (3.2) is satisfied will be called *potentially safe* and the system will be said *potentially stable* under $\mathbf{Q}$, where the adverb "potentially" is meant to account for the condition (3.2) being only necessary.

The set of all potentially safe loads will be denoted by K, whose definition following from condition (3.2) can be stated in the form:

$$\mathbf{Q} \in \mathrm{K} \Leftrightarrow \begin{cases} \exists\, \underline{\underline{\sigma}} \text{ statically admissible (s.a.) with } \mathbf{Q} \\ \underline{\underline{\sigma}}(\mathbf{x}) \in G(\mathbf{x})\ \forall \mathbf{x} \in \Omega. \end{cases} \tag{3.3}$$

The properties of K in the load-space $\{\mathscr{R}^n\}$ are derived from the properties of the domains $G(\mathbf{x})$:

$\mathbf{Q} = 0$ is potentially safe,

K is star-shaped with respect to $\mathbf{Q} = 0$,

K is convex if all $G(\mathbf{x})$ are convex.

The loads at the boundary of K in $\mathscr{R}^n$ are called the *extreme loads* for the system, corresponding to the fact that any load outside K is certainly unsafe.

The construction of K aims at determining the extreme loads $\mathbf{Q}^*$. The very definition (3.3) provides the so-called *static approach* approaching K "from inside" by constructing points within K.

This construction will be simplified by taking advantage of the star-shape of K, or of its convexity when valid, as shown in Figure 5.

It is worth noticing that in the (frequent) case of a single (positive) load parameter the static approach and the construction just exposed will lead to lower bounds for the extreme value Q^* of Q.

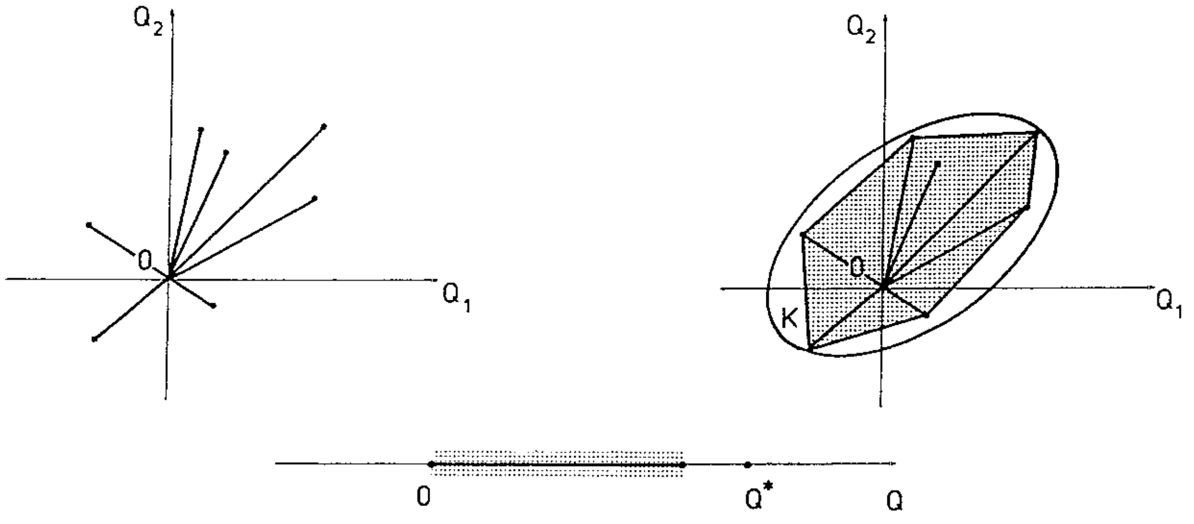

Fig. 5 — Static approach "from inside" for a star-shaped domain and a convex domain and in the case of one load parameter

3.3. KINEMATIC APPROACH "FROM OUTSIDE"

Starting from this static approach the principle of virtual work makes it possible to derive a formulation based upon the construction of kinematically admissible (k.a.) velocity fields for the determination of K.

Within the formalism of load parameters the principle of virtual work applied to problem of statics become:

(3.4)
$$\begin{gathered}\forall \underline{\underline{\sigma}} \text{ s.a. with } \mathbf{Q}(\underline{\underline{\sigma}}),\\ \forall \mathbf{U} \text{ k.a. with } \dot{\mathbf{q}}(\mathbf{U}),\\ \int_{\Omega} \underline{\underline{\sigma}}(\mathbf{x}) : \underline{\underline{\mathbf{d}}}(\mathbf{x})\, d\Omega + \int_{\Sigma} [\![\mathbf{U}(\mathbf{x})]\!] \cdot \underline{\underline{\sigma}}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, d\Sigma = \mathbf{Q}(\underline{\underline{\sigma}}) \cdot \dot{\mathbf{q}}(\mathbf{U})\end{gathered}$$

with $\underline{\underline{\sigma}}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = \sigma_{ij}(\mathbf{x})\, n_j(\mathbf{x})$ and $\underline{\underline{\sigma}}(\mathbf{x}) : \underline{\underline{\mathbf{d}}}(\mathbf{x}) = \sigma_{ij}(\mathbf{x})\, \mathrm{d}_{ji}(\mathbf{x})$ (summation on repeated subscripts) which provides the dual formulation of the s.a. condition in Eq. (3.3).

In order to take into account the conditions imposed by the strength capacities of the material the following "π functions" will be introduced depending on the point $\mathbf{x}$ considered in Ω and on the local values of $\underline{\underline{\mathbf{d}}}$, or $[\![\mathbf{U}]\!]$ following $\mathbf{n}(\mathbf{x})$, at the point:

(3.5)
$$\pi(\mathbf{x}; \underline{\underline{\mathbf{d}}}(\mathbf{x})) = \operatorname{Sup}\{\underline{\underline{\sigma}}(\mathbf{x}) : \underline{\underline{\mathbf{d}}}(\mathbf{x}) \mid \underline{\underline{\sigma}}(\mathbf{x}) \in G(\mathbf{x})\}$$

(3.6)
$$\pi(\mathbf{x}; \mathbf{n}(\mathbf{x}), [\![\mathbf{U}(\mathbf{x})]\!]) = \operatorname{Sup}\{[\![\mathbf{U}(\mathbf{x})]\!] \cdot \underline{\underline{\sigma}}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \mid \underline{\underline{\sigma}}(\mathbf{x}) \in G(\mathbf{x})\}$$

since $\underline{\underline{\sigma}}(\mathbf{x}) = 0$ is allowable both π functions are positive.

They represent the *maximum resisting work* (or power) which can be developed at point $\mathbf{x}$ in the strain rate $\underline{\underline{\mathbf{d}}}(\mathbf{x})$ or in the velocity jump $[\![\mathbf{U}(\mathbf{x})]\!]$ following $\mathbf{n}(\mathbf{x})$, allowing for the restriction on $\underline{\underline{\sigma}}(\mathbf{x})$ due to the strength capacities.

For any (virtual) velocity field $\mathbf{U}$ the maximum resisting work is expressed by the functional $P(\mathbf{U})$:

$$P(\mathbf{U})=\int_{\Omega}\pi(\mathbf{x};\,\underline{\underline{\mathbf{d}}}(\mathbf{x}))\,d\Omega+\int_{\Sigma}\pi(\mathbf{x};\,\mathbf{n}(\mathbf{x}),\,[\![\mathbf{U}(\mathbf{x})]\!])\,d\Sigma \tag{3.7}$$

which leads to the following statement derived from definition (3.3) and valid for any potentially safe load:

$$\begin{gathered}\forall \mathbf{Q}\in K,\\ \forall \mathbf{U}\text{ k.a. and corresponding } \dot{\mathbf{q}}(\mathbf{U})\\ \mathbf{Q}.\dot{\mathbf{q}}(\mathbf{U})-P(\mathbf{U})\leqq 0.\end{gathered} \tag{3.8}$$

This result is the basis of the approach of K from the outside as sketched in figure 6 since it implies that K is included in the half-space defined from the (virtual) kinematically admissible velocity field $\mathbf{U}$ by the inequality (3.8) considered with respect to $\mathbf{Q}$ in $\mathscr{R}^n$, so that:

$$\begin{gathered}\forall \mathbf{U}\text{ k.a. and corresponding } \dot{\mathbf{q}}(\mathbf{U})\\ K\subset\{\mathbf{Q}.\dot{\mathbf{q}}(\mathbf{U})-P(\mathbf{U})\leqq 0\}\end{gathered} \tag{3.9}$$

It leads to an upper bound for the extreme load Q^* in the case of one (positive) load parameter:

$$\begin{cases}\forall \mathbf{U}\text{ k.a. with } \dot{q}(\mathbf{U})>0\\ Q^*\leqq P(\mathbf{U})/\dot{q}(\mathbf{U}).\end{cases} \tag{3.10}$$

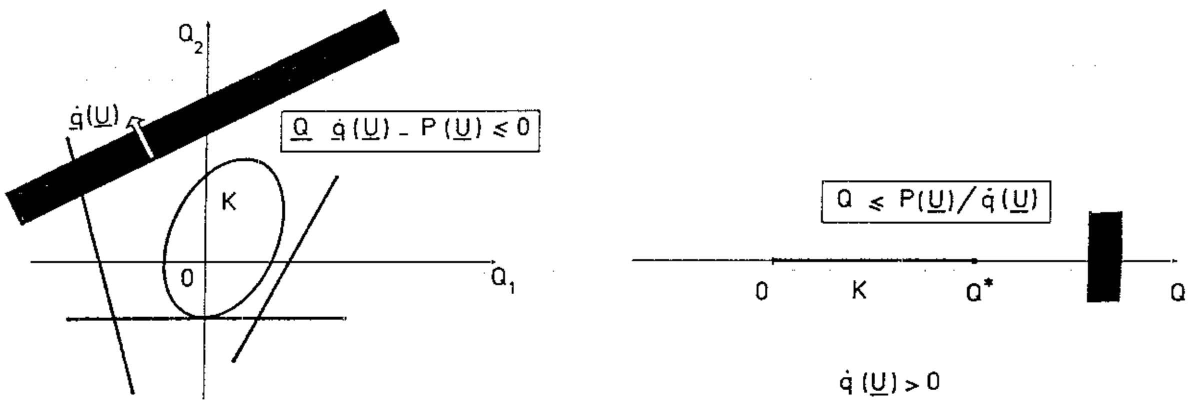

Fig. 6. – Kinematic approach "from outside".

In the case when all $G(\mathbf{x})$ are convex and under complementary assumptions it has been proved ([Nayroles, 1970]; [Frémond & Friaà, 1978]; [Friaà, 1979]) that the exact dual definition of K is given by Eq. (3.9), meaning that K can be generated by applying Eq. (3.9) to all k.a. virtual velocity fields.

It is worth pointing out that the fundamental idea of the kinematic approach, that is defining the strength domain by duality through the π functions, is apparent in a paper by Prager [1955].

3.4. COMMENTS

As mentioned earlier the terminology "potentially" aims at precisely stating the significance of the results obtained: more information about the behaviour of the constitutive material would be necessary to draw a stronger conclusion as regards stability rigorously (which can be done for instance in the case of an elastic perfectly plastic material under the assumption of the principle of maximum plastic work). Yet it may appear as reducing the impact of the theory if not depreciating it, a feeling which must be corrected: as a matter of fact any yield-design approach undergoes the same limitation and should take the same care when stating stability results; this is true in particular for the methods often used for practical applications to stability analyses in soil mechanics and for the "safety" factors they provide.

Anyway, after having drawn attention to that point, the terminology will be simplified according to common usage by deleting the "potentially" term in all that follows.

4. Practical application of the theory of yield design to soil-mechanics problems

4.1. SOME GUIDELINES

Many practical problems encountered in soil-mechanics are currently analysed through approaches which proceed from the yield design theory. They range from the determination of the bearing capacity of footings (in various circumstances: axial or inclined loads, with tilting moments, in the vicinity of slopes, ...) to stability analyses of slopes, embankments or tunnels.... Determining the extreme loads for the corresponding properly modelled problems with one or several load parameters will provide estimates for the bearing capacities or for the safety factors within the framework of the theory of yield design and with the significance already discussed (v.z. sec. 3.4).

The results so obtained together with the data coming out of other approaches will contribute to the specifications of practise.

Some characteristic patterns of the yield design approaches can be recalled:

- "crudeness" of the constitutive model,
- accessibility of the thought process,
- intervention of intuitive considerations derived from daily experience when devising s.a. stress-fields or k.a. virtual velocity-fields, even if this has to be carefully managed to ensure mechanical consistency.

They participate in the efficiency of the method. Unless for "acedemic" purposes (which are not to be discarded either) the game will not be to seek stubbornly for the mathematically exact values of the extreme loads but to apply the static and kinematic approaches to derive significant lower and upper estimates at a "reasonable price".

4.2. STATIC APPROACH

When applying the static approach simple s.a. stress-fields will currently be used, such as piecewise linear fields in the case of constant bulk forces (for which it is worth reminding that, besides the partial differential field equations of equilibrium (2.8), static admissibility implies the continuity of the stress vector acting on any stress discontinuity surface).

Figure 7 recalls Drucker & Prager's [1852] static approach for the stability analysis of the vertical cut which leads to the lower estimate for $(\gamma h/C)$:

$$2\tan(\pi/4+\varphi/2) \leq (\gamma h/C)^* \tag{4.1}$$

For the problem as settled at Section 2.1 the proper load parameter would be γ and the corresponding extreme load γ^*. Through dimensional analysis the stability of the cut is proved to be governed by the dimensionless ratio $(\gamma h/C)$ as it appeared already in the study performed at Section 2.1: thence the notation adopted in Ineq. (4.1).

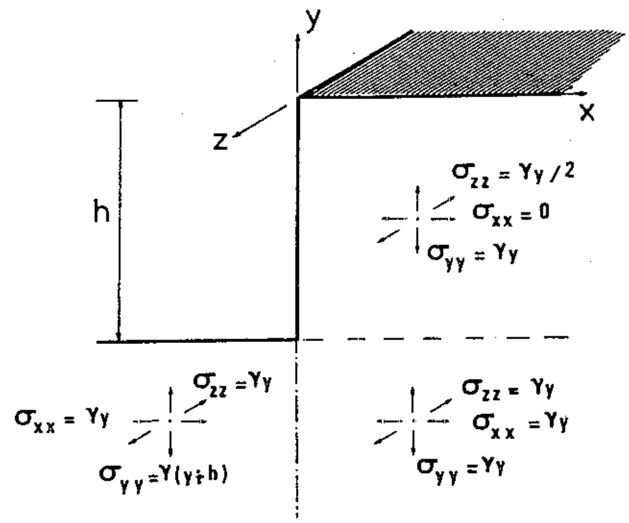

Fig 7. – Static approach for the problem of the vertical cut from [Drucker & Prager, 1952].

More sophisticated stress-fields may also be constructed, especially when dealing with plane problems where the mathematical method of characteristics has been extensively used (e.g. [Massau, 1899]; [Mandel, 1942]; [Hill, 1950]; [Sokolovski, 1955, 1960, 1965]; [Salençon, 1974]; ....).

Anyway, whatever the sophistication of the method it is essential to bear in mind that the results obtained will be of no significance if the stress field has not been shown to comply with the strength capacities of the constitutive soil, in the whole system under consideration ([Bishop, 1953]; [Salençon, 1969]).

4.3. Kinematic approach

Before dealing with the practical implementation of the kinematic approach, some consideration should be given to Eq. (3.8) or (3.9). It turns out that for Eq. (3.9) to be effective the following two conditions should be satisfied:

$$\dot{\mathbf{q}}(\mathbf{U}) \neq 0 \tag{4.2}$$

$$P(\mathbf{U}) < +\infty \tag{4.3}$$

The first one means that the relevant virtual velocity fields to be retained should cause the external forces to work (which is quite in accordance with common sense!).

The second condition together with Eq. (3.7), notwithstanding some mathematical reservations, implies that the virtual velocity field should be chosen so that

$$\pi(\mathbf{x};\, \underline{\underline{\mathrm{d}}}(\mathbf{x})) \text{ be finite everywhere } \Omega \tag{4.4}$$

and

$$\pi(\mathbf{x};\, \mathbf{n}(\mathbf{x}),\, [\![\mathrm{U}(\mathbf{x})]\!]) \text{ be finite everywhere in } \Sigma, \tag{4.5}$$

otherwise the upper estimate so obtained would be infinite. Though being but one aspect of the minimizing process to be performed in order to get as good an approach from outside as possible, this condition deserves particular attention as it will rule the choice of the relevant virtual velocity fields from the beginning.

Two examples will be hereafter considered.

4.4. Stability of a vertical cut for a purely cohesive soil

For a purely cohesive soil the strength criterion is written in the particular form of Eq. (2.7) where $\varphi=0$:

$$f(\underline{\underline{\sigma}}(\mathbf{x})) = \operatorname*{Sup}_{i,\, j=1,\, 2,\, 3} \{\sigma_i - \sigma_j - 2\,\mathrm{C}\} \leqq 0. \tag{4.6}$$

The corresponding π functions have the following expressions (*cf.* [Salençon, 1983]):

$$\begin{cases} \pi(\underline{\underline{\mathrm{d}}}) = +\infty & \text{if } \operatorname{tr}\underline{\underline{\mathrm{d}}} = \mathrm{d}_1 + \mathrm{d}_2 + \mathrm{d}_3 \neq 0 \\ \pi(\underline{\underline{\mathrm{d}}}) = \mathrm{C}(|\mathrm{d}_1| + |\mathrm{d}_2| + |\mathrm{d}_3|) & \text{if } \operatorname{tr}\underline{\underline{\mathrm{d}}} = 0 \end{cases} \tag{4.7}$$

and

$$\begin{cases} \pi(\mathbf{n},\, [\![\mathrm{U}]\!]) = +\infty & \text{if } [\![\mathrm{U}]\!].\mathbf{n} \neq 0 \\ \pi(\mathbf{n},\, [\![\mathrm{U}]\!]) = \mathrm{C}\,|[\![\mathrm{U}]\!]| & \text{if } [\![\mathrm{U}]\!].\mathbf{n} = 0. \end{cases} \tag{4.8}$$

This implies that the *relevant* k.a. virtual velocity fields to be retained when applying the kinematic approach *with that material* should induce *no volume change* ($\operatorname{tr}\underline{\underline{\mathrm{d}}}=0$) and that they can only exhibit *tangential velocity discontinuity* ($[\![\mathrm{U}]\!].\mathbf{n}=0$) that is the expression of the no volume change condition in terms of jumps. In that case the velocity jump surfaces will actually deserve the name of slip surfaces for the corresponding virtual velocity field.

Considering the problem of the vertical cut with the intention of applying the kinematic approach it appears through the same arguments as already presented in Section 2 that the analysis can be restricted to a slice of arbitrary thickness D parallel to plane Oxy with no resisting forces nor work developed on the parallel sections at both extremities. (A detailed presentation of this type of yield design problems—*plane strain yield design problems*—will be found in [Salençon, 1983] where it is shown how they can be studied through a two-dimensional yield design analysis)

For the slice of thickness D the following k.a. virtual velocity field can be retained in accordance with Eq. (4.7) and (4.8) and conditions (4.4) and (4.5): the prismatic volume OABB'A'O' is given a virtual downwards translation motion with the velocity **V** parallel to AB, while the rest of the soil mass remains motionless (*Fig.* 8).

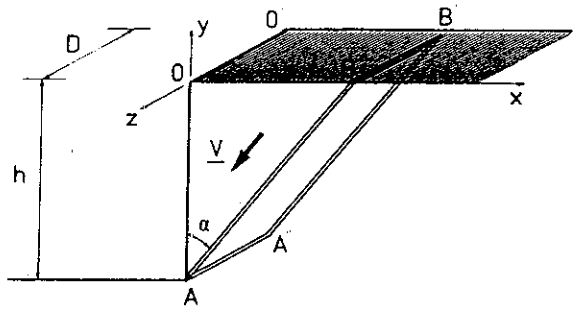

Fig. 8. — Stability of a vertical cut in a purely cohesive soil: kinematic approach.

The external work performed in this virtual velocity field by the gravity forces writes:

$$Q\dot{q}(U) = \frac{\gamma h^2 D}{2} V \sin\alpha \tag{4.9}$$

while the maximum resisting work being developed along the plane ABB'A' in the tangential velocity discontinuity reduces to:

$$P(U) = CV h D/\cos\alpha. \tag{4.10}$$

Thence through Ineq. (3.10)

$$(\gamma h/C)^* \leqq 4/\sin 2\alpha. \tag{4.11}$$

Looking for the minimum of the second hand of this upper bound with respect to α gives $\alpha = \pi/4$ and

$$(\gamma h/C)^* \leqq 4 \tag{4.12}$$

as the best upper estimate of $(\gamma h/C)^*$ for this class of one parameter k.a. virtual velocity fields.

The following comments can be made.

(*a*) Comparing the upper bound (4.12) with the value obtained in Section 2.1 (*Ineq.* 2.6) for this particular case where $\varphi = 0$ shows that both results are identical:

$$(\gamma h/C) \leqq 4$$

is proved a necessary condition for the stability of the cut.

(*b*) This is not a mere coincidence!

As a matter of fact the kinematic reasoning performed here with a uniform translation motion given to the volume OABB'A'O' results in checking the global equilibrium of that volume, as regards the resultant component in the direction of the motion, of the gravity forces in Ω and resisting forces developed on ABB'A'.

Through the use of function $\pi(\mathbf{n}, [\![\mathbf{U}]\!])$ here, the dual procedure in the kinematic approach

– automatically finds out the axis along which the global equilibrium equation will produce a non trivial necessary stability condition due to the expression of the strength condition (here this relevant axis is AB).

– selects the distribution(s) of the resisting forces which most favour(s) the global equilibrium of the volume so that external forces being not balanced under these conditions will necessarily mean the instability of the cut.

(*c*) One may question about the admissibility of the considered velocity field since the tip AA′ of volume OABB′A′O′ being given the velocity **V** parallel to AB would penetrate the horizontal surface of the motionless mass of the rest of the soil.

Many arguments have been put forward in order to solve this apparent paradox which occurs quite frequently in the applications of the kinematic approach: for those which are relevant they appear adequate to the problem under consideration with no versatility to be applied to similar cases, and do actually miss the fundamental point.

The answer lies in the very significance of virtual k. a velocity fields that must be looked at as *test functions*, the "mathematical tools" in the dualization procedure. This means that they are piecewise continuous with piecewise continuous derivatives, but no condition is imposed on the possible velocity discontinuity. Applying the kinematic approach we have seen that the "strength" of the material, through the corresponding $\pi(\mathbf{n}, [\![\mathbf{U}]\!])$ function imposed the choice of tangential velocity discontinuity on the plane AA′B′B for the value of $\pi(\mathbf{n}, [\![\mathbf{U}]\!])$ to remain finite. *No such condition is to be found to impose any constraint on* **U** *along* AA′ and this can be understood from the following: the three-dimensional continuum model does not take lineal densities of forces into account (nor does the two-dimensional model with punctual ones) and therefore the static analysis performed at section 2.1 did not consider any lineal resisting force at the vertex of volume (1) along AA′, which is consistent with no term appearing on AA′ in the dual formulation.

(*d*) Another result of the kinematic approach is to answer the question of whether the consideration of the moment equation of the global equilibrium of volume (1) in the static analysis performed at section 2.1 would have led to an additional limitation on $(\gamma h/C)$ for the stability necessary condition.

The answer is negative and is apparent from the dual formulation: considering the moment equation corresponds in the dual formulation to a rigid body rotational motion of volume OABB′A′O′ inducing a velocity discontinuity across the plane ABB′A′ which would not be tangential everywhere, unless the axis of rotation parallel to Oz be rejected at infinity in the direction normal to AB which is the case already examined.

This illustrates the fact that, but for that particular case which leads to the final result already obtained, the strength capacities of the soil will have no limiting consequence on the balance of the moment equation of global equilibrium of volume (1).

Thus Ineq. (4.12) expresses the best result to be derived from the consideration of the global equilibrium of triangular prismatic volumes.

(*e*) The same reasoning, through the kinematic approach as the dual formulation of the static analysis, shows that circular cylindrical volumes may also be considered for the "global equilibrium check" and produce significant results. In that case the relevant equation to provide a necessary stability condition on $(\gamma h/C)$ turns out to be the moment equation with respect to the geometrical axis of the cylinder (*Fig.* 9).

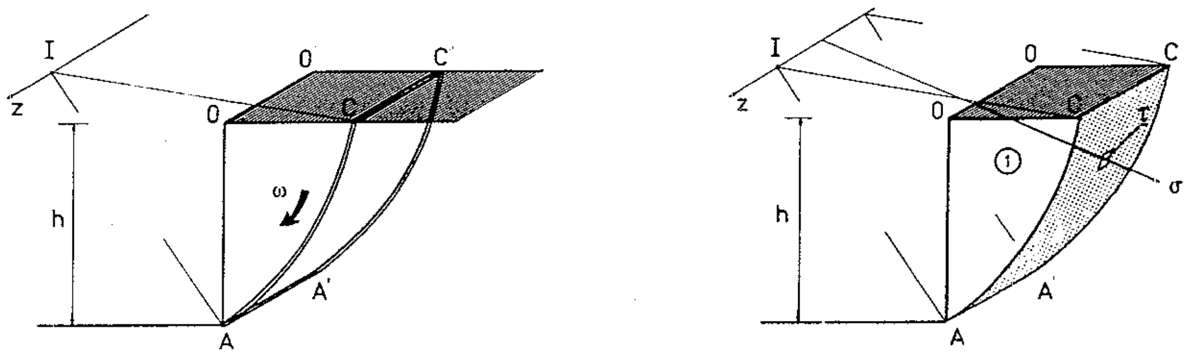

Fig 9. — Stability of a vertical cut in purely cohesive soil: kinematic approach with a rigid body rotational motion and "global equilibrium check".

The result obtained by looking for the strongest necessary condition over all cylindrical volumes is classical (v.z. [Taylor, 1937, 1948]) and is given by

$$(\gamma h/C)^* \leqq 3.83. \tag{4.13}$$

This upper estimate is better than condition (4.12); this is no surprise since the analyses with triangular prismatic volumes are but particular cases of the present ones.

(*f*) Finally it also appears from the dual formulation that no other cylindrical volumes can be considered significantly for a static analysis through the "global equilibrium check": they would not result in any further constraint on $(\gamma h/C)^*$.

4.5. Stability of a vertical cut for a soil exhibiting both cohesion and friction

For a cohesive and frictional soil the strength criterion has been written at Eq. (2.7). The corresponding π functions have the following expressions (*cf.* [Salençon, 1983]):

$$\begin{aligned} \pi(\underline{\underline{d}}) &= +\infty \quad \text{if} \quad \operatorname{tr}\underline{\underline{d}} < (|d_1|+|d_2|+|d_3|)\sin\varphi \\ \pi(\underline{\underline{d}}) &= \frac{C}{\tan\varphi}\operatorname{tr}\underline{\underline{d}} \quad \text{if} \quad \operatorname{tr}\underline{\underline{d}} \geqq (|d_1|+|d_2|+|d_3|)\sin\varphi \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} \pi(\mathbf{n}, [\![\mathbf{U}]\!]) &= +\infty \quad \text{if} \quad [\![\mathbf{U}]\!].\mathbf{n} < |[\![\mathbf{U}]\!]|\sin\varphi \\ \pi(\mathbf{n}, [\![\mathbf{U}]\!]) &= \frac{C}{\tan\varphi}[\![\mathbf{U}]\!].\mathbf{n} \quad \text{if} \quad [\![\mathbf{U}]\!].\mathbf{n} \geqq |[\![\mathbf{U}]\!]|\sin\varphi. \end{aligned} \tag{4.15}$$

The implications regarding the relevant k.a. virtual velocity fields are quite different from those derived in the case of a purely cohesive soil (*Sec.* 4.4).

From Eq. (4.14) it appears that when applying the kinematic approach with that material the k.a. virtual velocity fields to be retained *must induce volume changes*, exhibiting as a minimum the dilatancy given in Eq. (4.14). As regards the velocity discontinuities, Eq. (4.15) shows that for $\pi(\mathbf{n}, [\![\mathbf{U}]\!])$ to remain finite $[\![\mathbf{U}]\!]$ *cannot be purely tangential;* the velocity jump surfaces will therefore never properly deserve the name of slip surfaces; $[\![\mathbf{U}]\!]$ *must be inclined over the jump surface by an angle* β *at least equal to* φ and be directed outwards, corresponding to a virtual separation between both sides of the surface (*Fig.* 10).

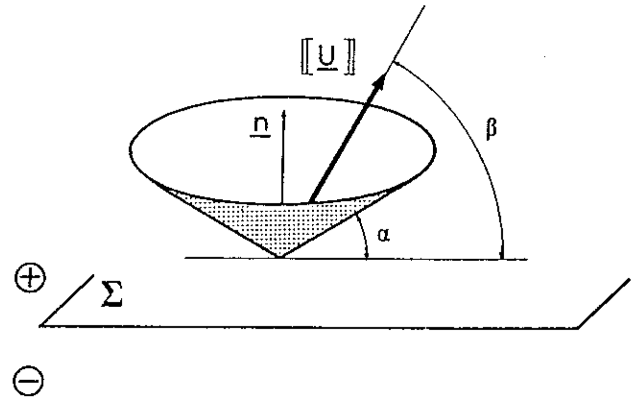

Fig. 10. – The velocity discontinuity $[\![\mathrm{U}]\!]$ of a relevant k.a. virtual velocity field.

For the slice of thickness D in the problem of the stability of the vertical cut the following k.a. virtual velocity field can be retained in accordance with Eq. (4.14) and (4.15) and conditions (4.4) and (4.5): the prismatic volume OABB′A′O′ is given a virtual downward translation motion with the velocity $\mathbf{V}$ inclined at an angle β to AB with $\varphi \leqq \beta \leqq \pi - \varphi$, while the rest of the soil mass remains motionless (*Fig.* 11).

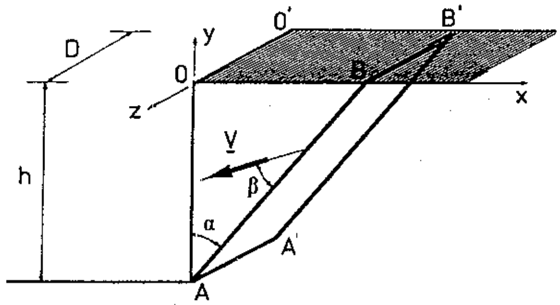

Fig. 11. – Stability of a vertical cut in a frictional and cohesive soil: kinematic approach

The external work performed in this velocity field by the gravity forces becomes:

$$\text{(4.16)} \qquad \mathrm{Q}\,\dot{\mathrm{q}}(\mathbf{U}) = \frac{\gamma h^2 \mathrm{D}}{2} \tan\alpha\, \mathrm{V} \cos(\alpha+\beta)$$

while the maximum resisting work being developed along the plane ABB′A′ in the velocity discontinuity is equal to

$$\text{(4.17)} \qquad \mathrm{P}(\mathbf{U}) = \frac{\mathrm{C}}{\tan\varphi}\, \frac{h\mathrm{D}}{\cos\alpha} \mathrm{V}\sin\beta.$$

It follows through Ineq. (3.10)

$$\text{(4.18)} \qquad \left(\frac{\gamma h}{\mathrm{C}}\right)^* \leqq \frac{2\sin\beta}{\tan\varphi \sin\alpha \cos(\alpha+\beta)}$$

where α and β are two parameters with the constraints

$$\left\{\begin{array}{l} 0 \leqq \alpha+\beta \leqq \pi/2 \\ \varphi \leqq \beta \leqq \pi-\varphi \end{array}\right. \tag{4.19}$$

Minimizing the second hand of Ineq. (4.18) with respect to β corresponds to

$$\beta=\varphi \tag{4.20}$$

for which Ineq. (4.18) becomes

$$(\gamma h/C)^* \leqq 2 \overset{\cos}{\cancel{\sin}}\, \varphi/\sin\alpha \cos(\alpha+\varphi) \tag{4.21}$$

[to be compared with Ineq. (2.5)].

Minimizing then with respect to α (as already done at *Sec.* 2.1) gives

$$(\gamma h/C)^* \leqq 4 \tan(\pi/4+\varphi/2) \tag{4.22}$$

as the best estimate of $(\gamma h/C)^*$ one may achieve for this class of two parameter k.a. virtual velocity fields.

This result also deserves some comments to complete those already made in the simpler case of the purely cohesive soil.

(*a*) As in the case of a purely cohesive soil it is found that the upper estimate derived from the kinematic approach with the virtual k.a. velocity fields of Figure 11 is equal to the result derived from the static analysis of Section 2.1. This comes out of the fact that, for any given volume OABB′A′O′ in Figure 11 the performed kinematic approach is exactly the dual formulation of the static analysis of paragraph (2.1) for the same volume (1) in Figure 4 dealing with the global equilibrium of volume (1) from the point of view of the resulting force.

(*b*) One may also notice that the optimality condition $\beta=\varphi$ in the kinematic approach means that, in the static analysis, the distributions of (σ, τ) along plane ABB′A′ which most favour the global equilibrium of volume (1) correspond to the equality

$$|\tau|=c-\sigma\tan\varphi \qquad \text{with} \quad \boldsymbol{\tau} \text{ in the direction } \mathbf{AB}$$

(which is quite evident here).

It also shows that the significant global equilibrium equation which is to provide the stability constraint on $(\gamma h/C)$ is the resulting force equation in the direction at angle φ to AB (the direction of **V**).

(*c*) The optimality condition $\beta=\varphi$ found in this particular case deserves being looked at in a more general perspective since the constraints on $[\![\mathbf{U}]\!]$ for the relevant virtual k.a. velocity fields derived from Eq. (4.15) are weaker than for a purely cohesive soil, namely

$$\varphi \leqq \beta \leqq \pi-\varphi. \tag{4.23}$$

The proofs of the statements to appear hereafter can be found in the results given in [Salençon, 1983, chap. VI]: they rely on the kinematic dual formulation of the "global equilibrium check" static analysis and can be generalized whatever the slope of the cut.

The idea would be to profit by the liberty left by condition (4.23) to perform the kinematic approach using more general piecewise rigid body motion velocity fields with any cross section for the cylindrical volume in motion. Such a rigid motion will be either a rotation around any axis Iz or a translation with any velocity **V** (*Fig.* 12).

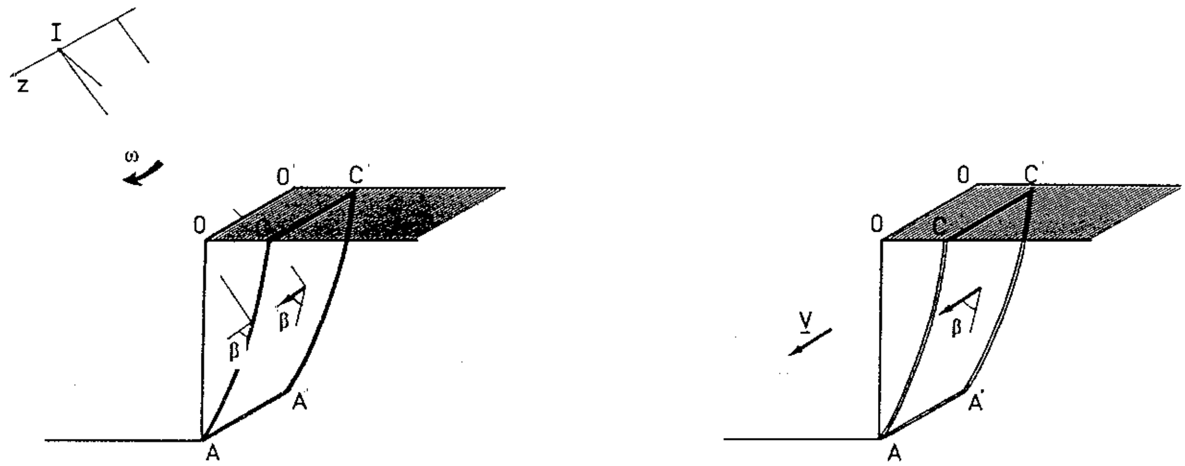

Fig. 12. – Stability of a vertical cut: more general kinematic approach using piecewise rigid body motion velocity fields.

From the static analysis of view this would amount to the "global equilibrium check" of a cylindrical volume (1) = OACC'A'O', with any cross section, being considered thoroughly.

For the *homogeneous* vertical cut the results are as follows.

Starting from the kinematic approach it comes out that:

• for a given axis Iz defining a rotation with a velocity ω (*Fig.* 13) the *most critical volume* OACC'A'O', that is the volume leading to the strongest constraint on $(\gamma h/C)^*$, is obtained when AC is an arc of a "φ" logspiral with the pole I so that the induced velocity discontinuity be inclined at the angle $\beta = \varphi$ to the surface ACC'A' at each point; such a "φ" logspiral is defined in polar coordinates (ρ, θ) attached to I by the following expression $\rho = \rho_0 \exp(-\theta \tan\varphi)$;

• as a particular case, when a translation is considered parallel to the velocity **V** the *most critical volume* OABB'A'O' is obtained when OAB is a triangle and **V** makes the angle φ with AB in the outwards direction (*Fig.* 13).

Consequently, from the "global equilibrium check" point of view it follows that:

• among all cylindrical volume (1) with any cross section being tested from the "global equilibrium check" standpoint, the *most critical* are those with a cross section defined by "φ" logspiral arc AC or a straight line AB (*Fig.* 14);

• for the first type, the significant equilibrium equation which leads to the necessary constraint on $(\gamma h/C)$ is the moment equation with respect to Iz where I is the pole of the considered logspiral;

• for the second type, the significant equilibrium equation is the resulting force equation in the direction inclined at the outward angle φ to AB in the plane Oxy (*Fig.* 14).

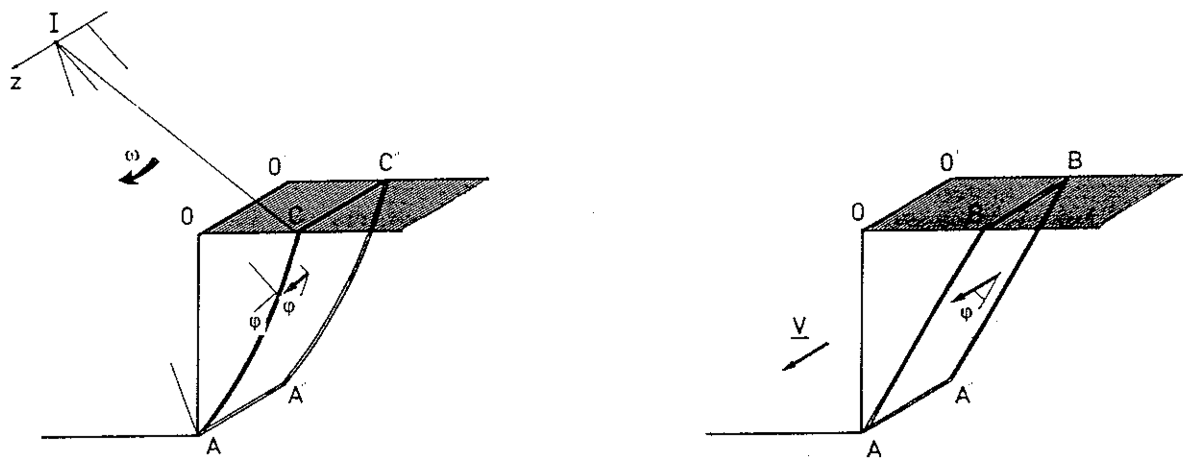

Fig. 13 – Stability of a vertical cut: most critical volumes for the kinematic approaches by piecewise rigid body motion velocity fields.

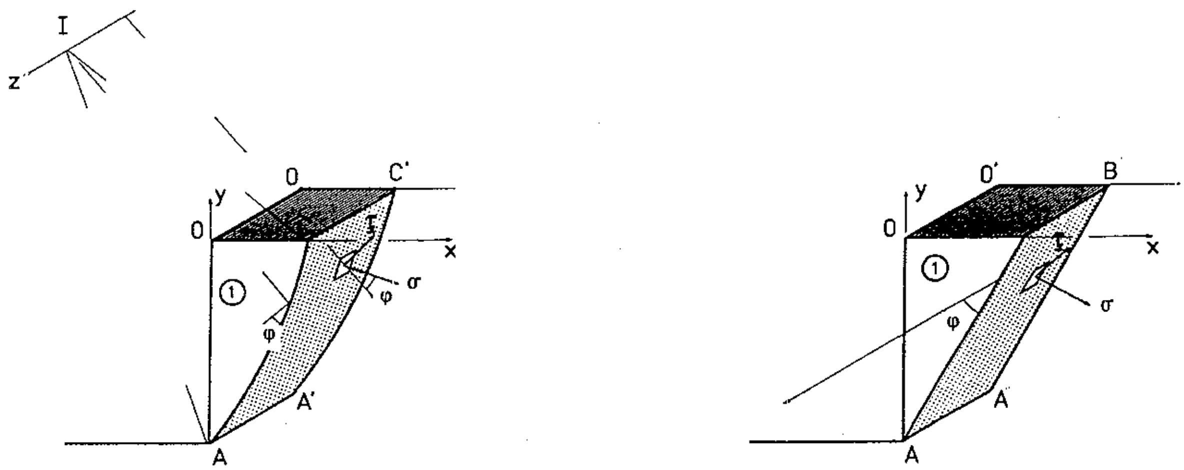

Fig. 14. – Stability of the vertical cut: most critical volumes for the kinematic "global equilibrium check".

The best result obtained through these equivalent analyses turns out to fit the formula [Chen, 1975]:

(4.24) $$(\gamma h/C)^* \leqq 3.83 \tan(\pi/4 + \varphi/2).$$

(*d*) the difference between Eq. (4.7) and (4.8) on one side and Eq. (4.14) and (4.15) on the other have led to conclusions concerning the general conditions to be fulfilled by relevant virtual k. a. velocity fields which might seem antinomic and make it difficult to imagine the possibility of passing continuously from one case to the other when $\varphi \to 0$.

This can be explained for instance for the relevant velocity discontinuities. For a frictional and cohesive soil ($\varphi \neq 0$, $C \neq 0$) it has been explained that the relevant $[\![\mathbf{U}]\!]$ must belong to the cone drawn in Figure 15. Making φ tend to zero, the allowable cone for $[\![\mathbf{U}]\!]$ becomes wider and flattens down into the upper half space (*Fig.* 15). At the same time it is seen from Eq. (4.15) where C is a constant that, keeping $|[\![\mathbf{U}]\!]|$ constant, the value of $\pi(\mathbf{n}, [\![\mathbf{U}]\!])$ tends to infinity when $\varphi \to 0$ unless $[\![\mathbf{U}]\!]$ belongs to the very boundary of the cone. It follows that, φ tending to zero, the relevant $[\![\mathbf{U}]\!]$ will tend to be confined in the velocity jump surface.

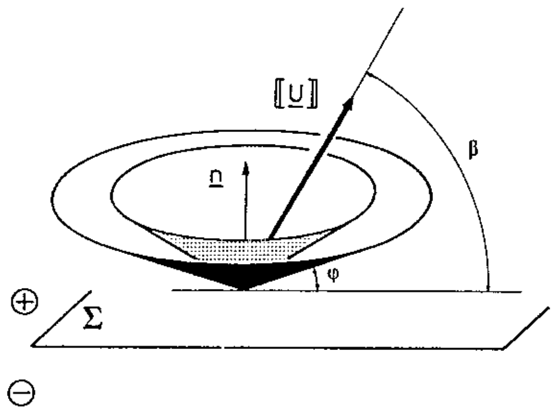

Fig 15 — The relevant velocity discontinuities as $\varphi \to 0$

4.6. A SHORT SURVEY OF THE IMPLEMENTATION OF THE KINEMATIC APPROACH

After the detailed study of those two examples it is useful to sum up some basic ideas about the application of the kinematic approach.

Simple virtual k. a. velocity fields $\mathbf{U}$ will be used.

Their choice is governed by the expressions of the π functions so that the maximum resisting work $P(\mathbf{U})$ remains finite.

For instance, piecewise rigid body motion velocity fields will be considered either in the form of Sections 4.4 and 4.5 where one rigid block is moving and the rest of the structure remains motionless or in more elaborate mechanisms where several blocks move with respect to one another (*Fig.* 16). In both cases the virtual velocity jumps will be governed by the condition that $\pi(\mathbf{n}, [\![\mathbf{U}]\!])$ remains finite as seen in Sections 4.4 and 4.5 and it is to be recalled that no paradoxical problems arise at vertices.

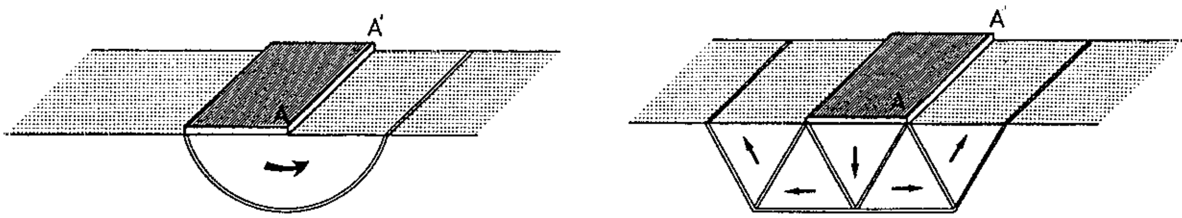

Fig. 16. — Bearing capacity of a strip footing on a purely cohesive soil: kinematic approaches by piecewise rigid body motion virtual k. a. velocity fields

The condition $\pi(\underline{\underline{\mathbf{d}}}) < +\infty$ will govern the relevant virtual k. a. velocity fields where $\underline{\underline{\mathbf{d}}} \neq 0$. For instance in the case of a purely cohesive soil the condition on $\mathbf{U}$ will be that $\operatorname{tr}\underline{\underline{\mathbf{d}}}=0$, that is "no volume change": such fields as simple shear velocity fields will be relevant for $\varphi=0$ soils, which will not be the case for soils exhibiting friction, due to Eq. (4.14).

When the problem under consideration accepts plane strain velocity fields as kinematically admissible the simplification due to passing from dimension 3 to dimension 2 makes it possible to construct more elaborate relevant virtual k. a. velocity fields.

Taking as an example the case of a purely cohesive soil ($\varphi=0$) and virtual plane strain k. a. velocity field parallel to Oxy, the "no volume change condition" results in generating the relevant virtual velocity fields in the following way:

Considering two families of mutually orthogonal curves (α- and β-lines), the components U_α and U_β of $\mathbf{U}$ at the current point verify the differential equations [Geiringer, 1937]

$$\begin{cases} dU_\alpha - U_\beta\, d\theta = 0 & \text{along the } \alpha\text{-lines} \\ dU_\beta + U_\alpha\, d\theta = 0 & \text{along the } \beta\text{-lines} \end{cases} \tag{4.25}$$

where θ is defined as $\theta=(Ox, \mathbf{e}_\alpha)$, $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$ being the unit vectors tangent to the α- and β-lines at the current point, $(\mathbf{e}_\alpha, \mathbf{e}_\beta)=+\pi/2$ (*Fig.* 17).

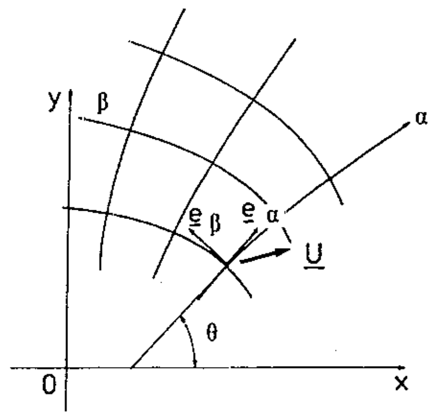

Fig. 17. – Plane strain relevant virtual k a. velocity fields for a purely cohesive soil: Geiringer's equations.

For a soil with friction, equations homologous to Eq. (4.25) can be established for the plane strain velocity fields which verify the equality condition in Eq. (4.14):

$$\mathrm{tr}\,\underline{\underline{d}} = \sin\varphi\,(|d_1|+|d_2|+|d_3|)$$

(*cf.* [S, 1974]).

A classical example of such a plane strain relevant virtual k. a. velocity field is recalled in Figure 18 where the volumes (1), (3), and (5) are given rigid body motions while in the volumes (2) and (4) the velocity is constant and orthoradial; the plane and cylindrical surfaces ABB'A', BCC'B', and CDD'C' and the symmetric ones are velocity discontinuity surfaces.

It may be added that, due to the corresponding simplifications, many yield design problems are often treated within the plane strain assumption, at least for a first study. As already said a detailed presentation as to the significance of this treatment within the framework of yield design may be found in [S, 1983] and many interesting solutions are available in numerous textbooks.

However it must be recalled once more that for a result to be significant from the yield design theory standpoint it is necessary that it be properly obtained through one

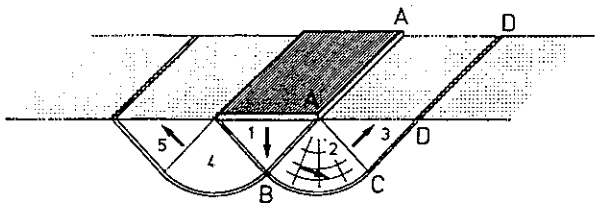

Fig 18. — Bearing capacity of a strip footing on a purely cohesive soil: Prandtl's velocity field

(or both) of the two presented approaches; whatever mathematical sophistication is no guarantee for this.

5. Conclusions

As an introduction to the application of the yield design theory to soil mechanics problems this paper does not claim to be comprehensive neither from the mathematical point of view which has been kept to a minimum nor for what concerns the numerous fields of applications and the particular methods involved. By choosing to thoroughly examine a classical example (the vertical cut stability) in order to introduce the various concepts it is hoped that the corresponding theoretical considerations, having received an illustration beforehand, will be more easily perceived.

Some main ideas will be recalled.

• The yield design theory relies on the necessary stability condition that "Equilibrium and Resistance should be Compatible". From this follows the *static approach*.

• The *kinematic approach* is but the mathematical dualization of the static one. The involved velocity fields are *virtual* velocity fields, meaning that they are the mathematical artefacts of this dualization. They do not claim any actuality as regards the true collapse modes of the considered system, even though experience seems sometimes to evidence some similarity between observed collapse modes and "good" virtual k. a. velocity fields.

• As apparent from the whole presentation, the yield design theory *does not rely on the so called rigid-plastic model*.

The possibilities of applications of the yield design in soil mechanics are really wide. Let some illustrative cases be finally mentioned.

• Stability analyses of seabed soils [Dormieux, 1989], of tunnels [Leca & Panet, 1988]; [Chambon & Corté, 1990]) etc.

• Stability analyses assuming *"tension cut off"* criteria for the constitutive soils, such as Tresca's or Coulomb's criteria with tension cut off, which means that no tensile stresses are allowed in the considered soil despite its exhibiting a cohesion. The corresponding expressions of the criteria and of the associated π functions are available in [S, 1983] as well as the same elements in the case of a limited but non zero resistance to tensile stresses.

• Reinforced soil structures stability analyses through methods combining the yield design and the *homogenization* theories. Results concerning this approach will be found for instance in ([de Buhan, 1986]; [Siad, 1987]; [de Buhan *et al.*, 1989]).

• Yield design study of reinforced soil structures through mixed modelling, that is adopting a continuum mechanics model for the constitutive soil and a beam (or a strip) model for the reinforcing inclusions. The interaction of the two models is mechanically consistent and the strength criteria and π functions are available. A computational program for the stability analysis and the design of reinforced soil structures (by soil nailing for instance) has been realized through the kinematic approach of yield design within this framework and turns out to be very efficient ([Anthoine, 1990]; [Anthoine & Salençon, 1989]). It is also worth noting that having a clear definition of the problem to be solved and of the mechanically consistent methods which the analyses rely on makes it easier to think over the necessary revision of the approach to safety.

Acknowledgement

The author thanks Pr. Willis for having drawn his attention recently onto the paper by Hill [1966] where the concept of extremal fields is studied based upon similar considerations of star-shaped and convex criteria.

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