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Yield design of reinforced earth walls by a homogenization method

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The increasing use of the reinforced earth technique in geotechnical engineering requires the development of reliable and practical yield design methods for reinforced earthworks. The method presented in this Paper originates from the idea that, from a macroscopic point of view, reinforced earth can be regarded as a homogeneous material with anisotropic properties, owing to the existence of privileged orientations due to the reinforcing strips. The strength criterion of such an equivalent material, which can be determined theoretically starting from the strength data of the reinforced earth components, turns out to be of the anisotropic frictional type. This criterion is applied to the stability analysis of a reinforced earth wall, making use of the yield design kinematic approach with rigid block failure mechanisms. The theoretical estimates obtained for the collapse height of the wall through this method appear to be in better agreement with experimental results than the values derived from classical design methods. Despite some limitations outlined in the Paper, the proposed yield design homogenization procedure may become an appropriate design method for reinforced soil structures in general.

KEYWORDS: reinforced earth; retaining walls; stability; failure; limit state design; anisotropy.

L'emploi de plus en plus répandu de la terre armée dans le domaine de la construction géotechnique, nécessite l'élaboration de méthodes de dimensionnement à la rupture des ouvrages en terre armée qui soient à la fois fiables et simples à utiliser. La méthode ici proposée part de l'idée selon laquelle la terre armée peut à l'échelle des ouvrages être considérée comme un matériau homogène mais anisotrope en raison de l'orientation privilégiée des armatures de renforcement. Le critère de résistance d'un tel matériau équivalent, déterminé par voie théorique à partir de ceux des constituants de la terre armée, apparaît comme du type 'frottant anisotrope'. Un tel critère est appliqué à l'analyse de stabilité d'un mur de soutènement en terre armée par une méthode cinématique de calcul à la rupture utilisant des mécanismes de rupture par blocs. Les estimations théoriques de la hauteur critique du mur que l'on obtient ainsi, se révèlent être en meilleur accord avec l'expérience que celles provenant des méthodes classiques de dimensionnement. En dépit de certaines limitations qui sont évoquées dans l'article, l'approche par homogénéisation en calcul à la rupture ainsi proposée est susceptible de constituer une méthode de dimensionnement efficace pour les ouvrages en sols renforcés.

NOTATION

φ	friction angle of the backfill soil
γ	specific weight of the backfill soil
H	height of the retaining wall
$0xy$	co-ordinate system
$0x$	direction parallel to the reinforcement
σ_{xx}, σ_{yy}, σ_{xy}	stress components
σ_1, σ_2	principal stresses
e	thickness of the reinforcements
ΔH	vertical spacing between two successive reinforcement layers
η	$e/\Delta H$, proportion by volume of the reinforcing material
C	cohesion of the reinforcing material
σ_0	$2\eta C$ tensile resistance of the reinforcing strips per unit transverse area
H^*	critical height of the retaining wall
K	$\gamma H/\sigma_0$ stability factor
K^*	$\gamma H^*/\sigma_0$
ω	angular velocity of the rotating block
Ω	centre of rotation
AB	velocity discontinuity line (arc of logspiral)
V	velocity jump across AB
ψ	angle made by V with the tangent to AB
(r, θ)	polar co-ordinates with respect to point Ω

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W_e work of the weight of the block OAB
W_r maximum resisting work
K^M upper bound estimate of K^*
K_p $\tan^2(\pi/4 + \varphi/2)$
C_{iso} $(K_p)^{1/2}\sigma_0/2$
H^*_{iso} critical height of the wall when modelling reinforced earth as an isotropic Mohr Coulomb material

INTRODUCTION

Since the original concept developed by Vidal in the early 1960s, reinforced earth has become widespread throughout the world as an economically suitable soil improvement technique for the building of earthworks. However, if the technological aspects relative to the use of this new composite material are now satisfactorily mastered, there remains a special need for reliable design methods aimed at rational prediction of the actual performance of reinforced earth structures.

Therefore, during the past two decades particular attention has been paid to the stability analysis of such structures, in close relation to the overall increase in strength of the soil that may be expected from the incorporation of the reinforcing elements. From this point of view, the first comprehensive theory of reinforced earth was set up by Schlosser & Vidal in 1969. Their investigation was mainly based on the idea that the tensile strength of reinforcements would give an apparent cohesion to the composite material through the frictional effects between soil and reinforcements. This initial conjecture has subsequently been largely corroborated by various experimental studies and theoretical analyses. Many references can be found in the general state of the art report established by Mitchell & Schlosser (1979) for the International Conference on Soil Reinforcement.

Stability analyses of reinforced earthworks have, up to now, always been directly adapted from the classical methods used for homogeneous soil structures, by trying for instance to take into account the resisting forces that the reinforcements are likely to develop along a potential failure surface. Unfortunately most of these approaches require complementary assumptions to be made concerning the interaction between the soil and the reinforcements, so that the calculations performed through these methods cannot be given any consistent theoretical interpretation.

The aim of this Paper is to propose a new design method for reinforced earth structures. This method relies on the yield design theory as generally stated by Salençon (1983) which has been applied to reinforced soils (de Buhan, 1986). It proceeds from the basic concept that, on the macroscopic scale, reinforced earth may be perceived as a homogeneous but anisotropic material, the strength criterion of which can be explicitly constructed given the strength characteristics of its components (soil and reinforcement). The yield design homogenization procedure derived from this concept is applied to the study of a reinforced earth wall. The theoretical results achieved through this method prove to be in good agreement with the experimental data.

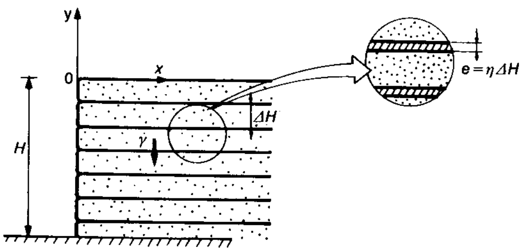

Fig. 1. Reinforced earth wall

STABILITY ANALYSIS OF REINFORCED EARTH RETAINING STRUCTURE

As an illustrative application of the theory presented in this Paper, the example of a reinforced earth vertical embankment, as shown in Fig. 1, is considered. It is assumed that the backfill soil is a dry cohesionless sand characterized by a friction angle φ and a specific weight γ. The horizontal reinforcements are taken to be continuous and long enough in the direction normal to the figure for the problem to be handled as a two-dimensional one, plane strain conditions being then fully satisfied. Referring to an $0xy$ coordinate system (where the $0x$-axis is taken as parallel to the reinforcements), the state of stress at any point (x, y) is described by three independent components: namely σ_{xx}, σ_{yy}, and σ_{xy}.

Consequently, reinforced earth may be modelled as a two-dimensional multilayered material. Denoting by e the thickness of the reinforcements and by ΔH the vertical spacing between two of them, the dimensionless ratio $\eta = e/\Delta H$ represents the voluminal fraction of the reinforcing material in the composite soil.

The strength properties of the soil are governed by a Coulomb failure condition with friction angle φ, which can be expressed as a function of the principal stresses (σ_1, σ_2) in the $0xy$ plane

$$\sigma_1 - \sigma_2 \leqslant (\sigma_1 + \sigma_2)\sin\varphi \quad \text{with} \quad \sigma_1 \geqslant \sigma_2 \quad (1)$$

where compressive stresses are counted positive, or, in terms of stress components

$$[(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2]^{1/2} \leqslant (\sigma_{xx} + \sigma_{yy})\sin\varphi \quad (2)$$

The constitutive material of the reinforcing layers is assumed to be a homogeneous purely cohesive medium, obeying a Tresca yield criterion with cohesion C, so

$$\sigma_1 - \sigma_2 \leqslant 2C \quad (3)$$

or

$$[(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2]^{1/2} \leqslant 2C \quad (4)$$

Perfect bonding between the soil and the reinforcements will be assumed, so no slipping phenomenon localized at their interface is considered in this analysis. According to the yield design theory (Appendix 1) for the reinforced earthwork to be safe under its own weight γ (acting as the only loading parameter, no external forces being applied to its boundary) there must exist a stress field inside the whole structure

(*a*) equilibrating γ and the stress boundary conditions
(*b*) and verifying the strength condition (2) or (4) at any point (x, y) depending on whether this point is located in the soil or in the reinforcements.

From the engineering standpoint, the problem is to evaluate the critical height H^* of the embankment, i.e. the maximum value of H for which the structure may be safe under γ

$$\text{stability of the structure} \Rightarrow H \leqslant H^*$$

Owing to the highly heterogeneous nature of the constitutive reinforced earth, many difficulties arise when trying to employ the yield design static or kinematic approaches directly. For that reason the implementation of the yield design methods for this type of structure usually requires simplifying assumptions to be made especially to model most effectively the reinforcing inclusions and their interaction with the surrounding soil. In most cases numerical methods have to be employed, which turn out to be far more complicated than for homogeneous earthworks. This is one of the major conclusions that can be drawn for the works of Ciss (1985), and Pastor, Turgeman & Ciss (1986) who used a finite element method to study the stability of a reinforced soil embankment within the framework of limit analysis. The problem may become intricate when a strength condition specific to the interface between the soil and the reinforcements has to be taken into account.

HOMOGENIZATION METHOD

The principle of the homogenization method (de Buhan & Salençon, 1983; Suquet, 1985; Salençon, 1984) is based on the concept that from a macroscopic point of view, i.e. as far as overall properties of the reinforced earth structure such as its collapse load are concerned, the constitutive reinforced soil can be perceived as a homogeneous medium. The strength capacities of this medium will be defined by means of a strength criterion, the macroscopic strength criterion of reinforced earth. The stability analysis of the initial heterogeneous structure then reduces to the investigation of an associated homogeneous problem (Fig. 2) obtained through this homogenization procedure. On account of the existence of preferential orientations due to the reinforcements, the homogenized material will exhibit anisotropic strength properties. The practical validity of such a homogenization procedure relies on the fulfilment of the following two conditions.

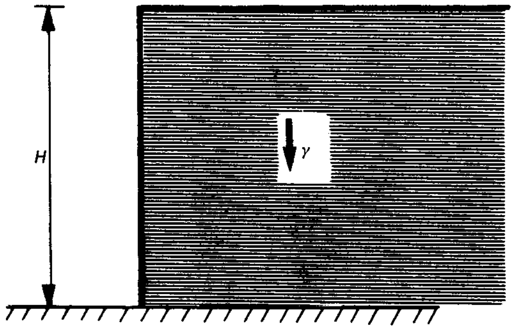

Fig. 2. Associated homogeneous retaining wall

(*a*) The reinforcing inclusions must be placed into the backfill soil to follow a regular pattern, so that from the fundamental point of view of continuum mechanics, reinforced earth can be regarded as a periodically heterogeneous medium.
(*b*) The characteristic scale of the reinforcement (i.e. the vertical spacing ΔH between two successive layers) must be small enough when compared to the total height of the embankment.

Under these conditions, and provided the appropriate definition of the macroscopic strength criterion is adopted, the stability analysis of the reinforced earth wall can be performed, in applying yield design approaches to the associated homogeneous structure.

Formulation of macroscopic strength criterion for reinforced earth

Starting from the two-dimensional multilayer model of reinforced earth (de Buhan, Salençon & Siad, 1986) as described in the section on stability analysis, it can be shown that the adequate definition of its macroscopic strength criterion is given

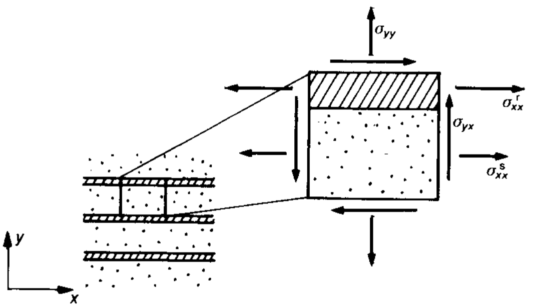

Fig. 3. Multilayer model of reinforced earth

by the set of allowable stresses σ_{xx}, σ_{yy} and σ_{xy} such that σ_{xx} may be split in the form

$$\sigma_{xx} = (1 - \eta)\sigma^s_{xx} + \eta\sigma^r_{xx} \tag{5}$$

with the stresses σ^s_{xx}, σ_{yy} and σ_{xy} and σ^r_{xx}, σ_{yy} and σ_{xy} satisfying the strength conditions of the soil (equation (2)) and of the reinforcing material (equation (4)) respectively (Fig. 3).

In the case of reinforced earth, the inclusion layers appear to be very thin metal strips, so that the volume ratio $\eta = e/\Delta H$ is very small (about 10^{-2} or 10^{-3}), while the shear strength C of the constitutive reinforcing material is far greater than the typical values usually observed for soils. Mathematically, this particular configuration may be derived from the above multilayered model as the limit case obtained by making η tend to zero and keeping $2\eta C$ equal to a constant σ_0: $\eta \to 0$, with $2\eta C = \sigma_0$ a constant, from which

$$\sigma_0 = 2\eta C = 2Ce/\Delta H = R_t/\Delta H \tag{6}$$

where R_t denotes the tensile strength of the reinforcing layers per unit length along the direction normal the $0xy$ plane (therefore the dimension of σ_0 is that of a stress).

The expression of the macroscopic strength criterion can then be greatly simplified in this case, and reduces to (Appendix 2)

$$[(\sigma_{xx} - \sigma_{yy} - \sigma)^2 + 4\sigma^2_{xy}]^{1/2} \leqslant (\sigma_{xx} + \sigma_{yy} - \sigma) \sin\varphi \tag{7}$$

with σ a parameter satisfying $|\sigma| \leqslant \sigma_0$. This result deserves some comments.

Definition (7) of the macroscopic strength condition of reinforced earth gives clear evidence of the increase in strength of the soil due to the introduction of reinforcing layers. Any state of stress that respects the strength condition in equation (2) of the soil alone, will satisfy the strength criterion in equation (7).

The theoretical framework adopted leads to a formulation of the macroscopic strength criterion closely connected with those previously conjectured by McLaughlin (1972) about fibre reinforced materials and more specifically by Sawicki (1983) and Sawicki & Leśniewska, (1987) concerning the macroscopic yield behaviour of reinforced earth. The main assumption of these authors was that the reinforcing inclusions were just acting inside the soil as tensile load carrying elements, offering no resistance to shear and bending loadings. This heuristic assumption appears fully justified by the homogenization approach.

It is possible within the scope of the yield design homogenization theory to allow for failure of the reinforcing strips by compressive buckling, which would occur when the parameter σ reaches the value $k\sigma_0$, where k is a non-dimensional factor possibly ranging from 0 (no compressive strength assigned to the strips) to 1. As it is usually assumed that the strips do not contribute significantly as compressive elements, the value $k = 0$ will be adopted from now on, and condition $|\sigma| \leqslant \sigma_0$ will be replaced by $-\sigma_0 \leqslant \sigma \leqslant 0$ in equation (7).

In order to assess the practical validity of the theoretical model proposed for reinforced earth, a comparison with available experimental data has been made by Mangiavacchi and Pellegrini (1985) on the basis of triaxial tests performed on sand samples reinforced by regularly spaced aluminium sheets inclined at different angles (Long & Ursat, 1977). The agreement between the theoretical predictions derived from equation (7) and the observed results turns out to be perfect when the study is concerned with compressive stresses that are the only ones considered in classical triaxial tests (de Buhan & Salençon, 1987).

Representation of macroscopic strength criterion in Mohr plane

In anticipation of application of the yield design homogenization method to the analysis of the reinforced wall stability, a convenient geometrical representation of the macroscopic strength condition in equation (7), with $-\sigma_0 \leqslant \sigma \leqslant 0$ is now presented. Considering any oriented facet whose inner normal n is inclined at an angle α with respect to the $0x$ direction (Fig. 4), there may be drawn in the corresponding (σ_n, τ_n) plane the set of allowable stress vectors acting on this plane, i.e. such that according to classical relations

$$\begin{cases} \sigma_n = \sigma_{xx}\cos^2\alpha + \sigma_{yy}\sin^2\alpha + 2\sigma_{xy} \\ \qquad \times \sin\alpha\cos\alpha \\ \tau_n = (\sigma_{yy} - \sigma_{xx})\sin\alpha\cos\alpha + \sigma_{xy} \\ \qquad \times (\cos^2\alpha - \sin^2\alpha) \end{cases} \tag{8}$$

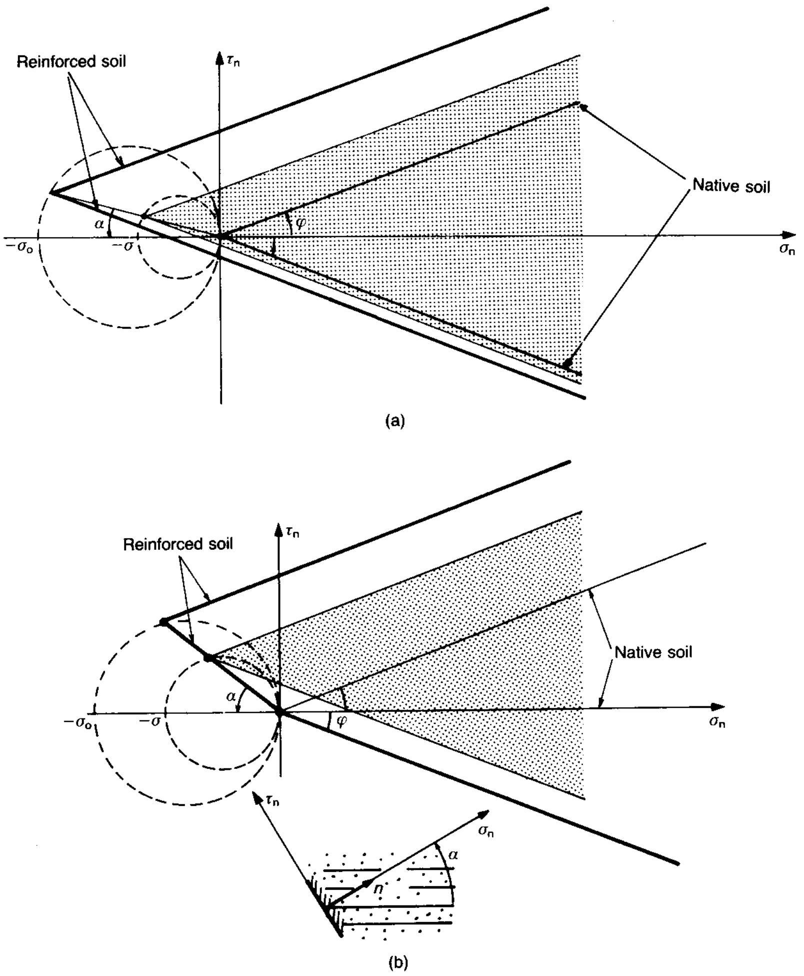

Fig. 4. Representation of macroscopic strength condition of reinforced earth in a (σ_n, τ_n) plane

where (σ_{xx}, σ_{yy}, σ_{xy}) verify (7). Introducing

$$\sigma'_{xx} = \sigma_{xx} - \sigma, \quad \sigma'_{yy} = \sigma_{yy}, \quad \sigma'_{xy} = \sigma_{xy}$$

it follows from equation (8)

$$\begin{cases} \sigma_n = \sigma'_n + \sigma \cos^2 \alpha \\ \tau_n = \tau'_n - \sigma \sin \alpha \cos \alpha \end{cases} \tag{9}$$

where σ'_n and τ'_n denote the normal and tangential components acting on the plane induced by the stress state (σ'_{xx}, σ'_{yy}, σ'_{xy}); as can be seen from equation (7) it satisfies the classical Coulomb failure condition (equation (2)), and therefore

$$|\tau'_n| \leqslant \sigma'_n \tan \varphi. \tag{10}$$

These relations make it possible to construct the macroscopic strength domain of reinforced earth in the (σ_n, τ_n) plane. Given any value of parameter σ lying between $-\sigma_0$ and 0, substitution for σ'_n and τ'_n from equation (9) into inequality (10) yields

$$|\tau_n + \sigma \sin \alpha \cos \alpha| \leqslant (\sigma_n - \sigma \cos^2 \alpha) \tan \varphi. \tag{11}$$

It follows from this last relation that the boundary of the macroscopic strength domain in the (σ_n, τ_n) plane can be drawn simply as the convex envelope of the classical Coulomb intrinsic failure curve (made up of two straight lines inclined at $\pm\varphi$ to the σ_n-axis and intersecting at the origin) and of the curve derived from that one through the translation defined by the vector of components ($-\sigma_0 \cos^2 \alpha$, $\sigma_0 \sin \alpha \cos \alpha$) as displayed in Fig. 4 where two such curves have been drawn which correspond to the cases $0 \leqslant \alpha \leqslant \varphi$ and $\varphi \leqslant \alpha \leqslant \pi/2$.

The anisotropy of reinforced earth as a homogeneous material is shown through this geometrical representation. Any isotropic strength criterion (such as the Coulomb criterion) would be represented by the same curve whatever the orientation α of the facet, which is not the case here.

The concepts of cohesion and friction angle, as usually introduced for isotropic or even purely cohesive anisotropic soils (Salençon & Tristán-López, 1981; Salençon, 1984) are no longer relevant to account for the anisotropic characteristics of the criterion so obtained. Nor can this anisotropy be described through the criteria proposed by Boehler & Sawczuk (1970) based on a generalization of the classical Coulomb failure condition using the notion of anisotropy tensor. Consequently the formulation of such an anisotropic criterion turns out to be quite original.

THEORETICAL ANALYSIS OF REINFORCED EARTH WALL STABILITY USING HOMOGENIZATION APPROACH

In considering the initial problem of the reinforced earth wall, dimensional analysis arguments will indicate that the critical height H^* of the wall will be

$$H^* = \frac{\sigma_0}{\gamma} K^*(\varphi) \tag{12}$$

where K^* is a non-dimensional factor, named the stability factor of the earthwork, which proves to be a function of angle φ alone. The aim of the forthcoming investigation is to obtain an upper bound estimate of H^* (or K^*) by means of the yield design kinematic approach (Appendix 1) applied to the previously defined homogenized structure, making use of the well-known rigid block failure mechanisms of the wall.

Such a failure mechanism can be considered, in which a block OAB rotates about point Ω with an angular velocity ω (Fig. 5) while the rest of the structure is kept motionless, so that AB is a velocity discontinuity line passing through the toe of the embankment. Denote by $\boldsymbol{V}$ the velocity jump vector when crossing AB following its normal $\boldsymbol{n}$ at point M which will be assumed to make a constant angle ψ with the tangent to AB. The latter is therefore an arc of a logspiral whose equation in a polar coordinate system (r, θ) with origin at Ω is

$$r = r_0 \exp\left[(\theta - \theta_0) \tan \psi\right] \tag{13}$$

where (r_0, θ_0) are the polar co-ordinates of point A. The associated velocity field (which is kine-

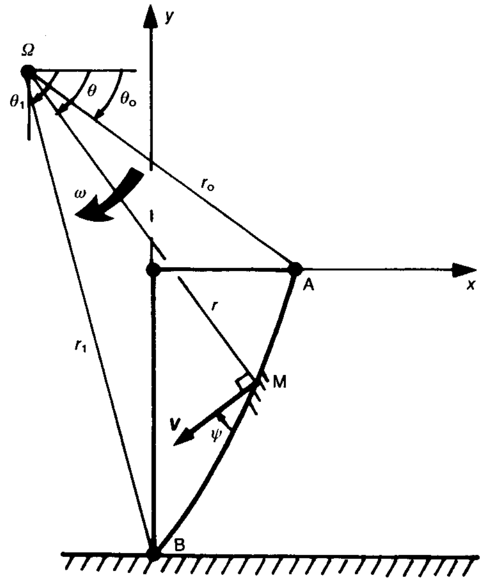

Fig. 5. Rotational failure mechanism of homogenized wall

matically admissible) is then completely defined by three angular parameters θ_0, θ_1, and ψ which characterize the geometry of the block and by the velocity parameter ω. For the sake of convenience all those parameters as well as θ and related angles will be counted positive clockwise in the calculations.

Now the yield design kinematic (or upper bound) theorem states that, given any such kinematically admissible failure mechanism, for the structure to be safe, i.e. for $\gamma H/\sigma_0 \leqslant K^*(\varphi)$, the work done by the external forces (the specific weight γ in the present case) must remain lower than or equal to the maximum resisting work calculated in the same mechanism.

Work done by external forces

The work done by external forces reduces to the work developed by the weight of the rotating block OAB, that is, referring to $0xy$ axes (where $0y$ is orientated upwards)

$$W_e = \int_{\mathrm{OAB}} -\gamma v_y \, \mathrm{d}x \, \mathrm{d}y \quad (\gamma > 0)$$

where v_y is the vertical component of the velocity at any point (x, y) of the block; or

$$W_e = \omega\gamma \int_{\mathrm{OAB}} (x + r_1 \cos \theta_1) \, \mathrm{d}x \, \mathrm{d}y = M_w \omega$$

where M_w denotes the moment of the weight of the block OAB with respect to Ω, which through

Green's theorem may be written as an integral along curve AB

$$M_w = \gamma \int_{AB} - y(x + r_1 \cos \theta_1) \sin (\theta - \psi) \, ds$$

$$= \gamma \int_{\theta_0}^{\theta_1} r^2(r \sin \theta - r_0 \sin \theta_0) \times \frac{\cos \theta}{\cos \psi} \sin (\theta - \psi) \, d\theta$$

as certain geometrical relations exist: $x + r_1 \cos \theta_1 = r \cos \theta$; $-y = r \sin \theta - r_0 \sin \theta_0$; and $ds = r \, d\theta / \cos \psi$. From which, taking equation (13) into account is obtained

$$W_e = M_w \, . \, \omega = \gamma \omega r_0^3 (m_1 - m_2 - m_3) \qquad (14)$$

with

$$m_1 = \{(3 \tan \psi \cos \theta_1 + \sin \theta_1) \times \exp [3(\theta_1 - \theta_0) \tan \psi] - (3 \tan \psi \cos \theta_0 + \sin \theta_0)\} \times [3(1 + 9 \tan^2 \psi)]^{-1}$$

$$m_2 = \tfrac{1}{6}(\cos^2 \theta_0 - \cos^2 \theta_1 \times \exp [2(\theta_1 - \theta_2) \tan \psi]) \sin \theta_0$$

$$m_3 = \tfrac{1}{3}(\sin \theta_1 \exp [(\theta_1 - \theta_0) \tan \psi] - \sin \theta_0) \times \cos^2 \theta_1 \exp [2(\theta_1 - \theta_0) \tan \psi]$$

Evaluation of maximum resisting work

Since the strain-rate field associated with any rigid-block failure mechanism is such that no deformation occurs outside the discontinuity line AB, the maximum resisting work W_r can be written

$$W_r = \int_{AB} \pi^{hom}(\boldsymbol{n}; \boldsymbol{V}) \, ds \qquad (15)$$

in which function $\pi^{hom}(\boldsymbol{n}; \boldsymbol{V})$ is the maximum resisting work per unit length along the discontinuity line AB. Referring to Fig. 6, $\pi^{hom}(\boldsymbol{n}; \boldsymbol{V})$ is defined as the maximum value of

$$-\sigma_n |\boldsymbol{V}| \sin \psi - \tau_n |\boldsymbol{V}| \cos \psi$$

(which may be interpreted as the scalar product of the stress vector (σ_n, τ_n) by $-\boldsymbol{V}$ for (σ_n, τ_n) satisfying the homogenized strength criterion of reinforced earth, i.e. conditions (9) and (10) with $-\sigma_0 \leqslant \sigma \leqslant 0$ and $\alpha = \psi - \theta$ (angles are considered positive anticlockwise in the Mohr plane).

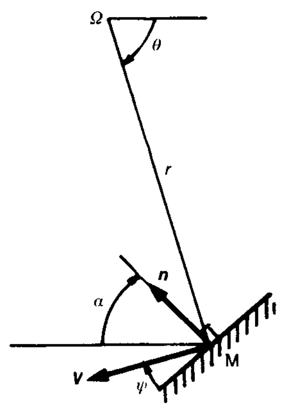

Fig. 6. Velocity jump at point (r, θ) of curve AB

The following equation is obtained (intermediate calculations are not reproduced here)

$$\pi^{hom}(\boldsymbol{n}; \boldsymbol{V}) = \begin{cases} \sigma_0 |\boldsymbol{V}| \langle \sin \theta \cos (\theta - \psi) \rangle^* \\ \quad \text{if} \quad \varphi \leqslant \psi \leqslant \pi - \varphi \\ +\infty \quad \text{otherwise} \end{cases} \qquad (16)$$

Fig. 7 displays a simple way to evaluate π^{hom} geometrically in the (σ_n, τ_n) plane. The expression of W_r is, for $\varphi \leqslant \psi \leqslant \pi - \varphi$, then

$$W_r = \omega \sigma_0 \int_{\theta_0}^{\theta_1} r^2 \langle \sin \theta \cos (\theta - \psi) \rangle \frac{d\theta}{\cos \psi}$$

that is, using equation (13) and integrating over θ from θ_0 to θ_1

$$W_r = \tfrac{1}{2} \omega \sigma_0 r_0^2 m_4 \qquad (17)$$

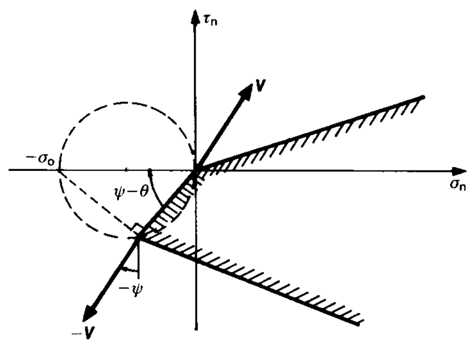

Fig. 7. Geometrical interpretation of function π^{hom}

* $\langle \cdot \rangle$ the positive part, i.e. $\langle x \rangle = x$ if $x \geqslant 0$, but otherwise $= 0$.

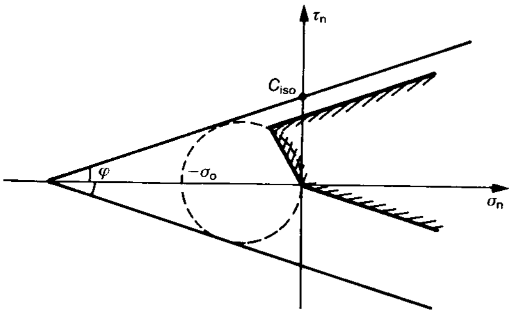

Fig. 8. Isotropic approximation of reinforced earth

with

$$m_4 = \sin^2\theta_1 \exp[2(\theta_1 - \theta_0)\tan\psi] + \begin{cases} -\sin^2\theta_0 + \cos^2\psi \\ \times \exp\left[2\left(\psi - \dfrac{\pi}{2} - \theta_0\right)\tan\psi\right] \\ \quad \text{if } \theta_0 \leqslant -\dfrac{\pi}{2} + \psi \\ 0 \quad \text{if } -\dfrac{\pi}{2} + \psi \leqslant \theta_0 \leqslant 0 \\ -\sin^2\theta_0 \quad \text{if } \theta_0 \geqslant 0 \end{cases}$$

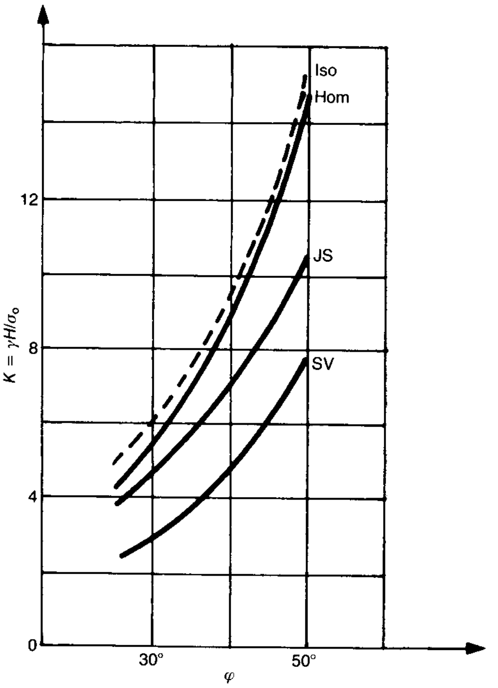

Fig. 9. Comparison between estimates of the stability factor K^* of a reinforced earth wall using different methods; SV—classical method (Schlosser & Vidal); $K = K_p$; JS—method proposed by Juran & Schlosser; Hom—Homogenization method ($K = K^M$); Iso—Isotropic approximation ($K \approx 1{\cdot}91\ K_p$)

Upper bound estimate of H^*

The yield design kinematic theorem applied to the homogenized structure leads to the following statement: given γ and σ_0, $W_e \leqslant W_r$ if $H \leqslant H^*$, whatever values of θ_1, θ_2, ψ and ω. From expressions (14) and (17) and assuming angular velocity ω to be positive, there is obtained $\gamma r_0^*/\sigma_0 \leqslant m_4/2(m_1 - m_2 - m_3)$ where $r_0^* = H^*/m_5$ and $m_5 = \sin\theta_1 \exp[(\theta_1 - \theta_0)\tan\psi] - \sin\theta_0$. Consequently

$$H^* \leqslant (\sigma_0/\gamma)K^* \leqslant (\sigma_0/\gamma)K^M \qquad (18)$$

with

$$K^M = \min_{\theta_0,\theta_1,\psi}\left[\frac{m_4 m_5}{2(m_1 - m_2 - m_3)}\right]$$
$$(\varphi \leqslant \psi \leqslant \pi - \varphi)$$

K^M is then determined through a numerical minimization procedure from the above analytical expression. It can be shown that this minimum is obtained for $\psi = \varphi$ which means that, with a restriction to rigid block failure mechanisms, the optimum, that is providing the best upper bound estimate for the critical height of the wall H^*, corresponds to an arc of a logspiral with angle φ.

COMPARISON WITH OTHER METHODS AND EXPERIMENTAL VALIDATION

Approximate calculation: reinforced earth as isotropic material

Another possible way to deal with the problem consists of making the assumption that reinforced earth can be modelled as a homogeneous Mohr Coulomb material with a friction angle φ, endowed with an equivalent isotropic cohesion $C_{iso} = (K_p)^{1/2}\sigma_0/2$, where $K_p = \tan^2(\pi/4 + \varphi/2)$ is the classical passive earth pressure coefficient, so that its uniaxial tensile strength in any direction is equal to σ_0. The corresponding strength domain in any (σ_n, τ_n) plane is bounded by two straight lines inclined at angles $\pm\varphi$ to the σ_n-axis and tangential to the dashed circle as sketched in Fig. 8. As it appears from this figure, such an isotropic approximation of reinforced earth as a homogenized material amounts to an overestimation of its strength characteristics, and therefore $H^* \leqslant H^*_{iso}$, where H^*_{iso} is the critical height of the isotropic homogenized retaining wall.

By using the same kinematic approach that involves rigid block failure mechanisms as in the case of the anisotropic homogenized wall, the classical upper bound solution is obtained

$$H^*_{iso} \leqslant H^M_{iso} \approx 3{\cdot}83(K_p)^{1/2}\frac{C^{iso}}{\gamma}$$

which provides immediately a simple upper bound estimate for H^*

$$H^* \leqslant H^*_{\rm iso} \leqslant H^{\rm M}_{\rm iso} \approx 1{\cdot}91 K_{\rm p}\sigma^0/\gamma$$

or which concerns the stability factor

$$K^* \leqslant 1{\cdot}91\ K_{\rm p}$$

Two classical design methods for reinforced earth walls

Figure 9 displays a comparison between different estimates of the critical height of the wall calculated with the following methods

(*a*) the homogenization approach as developed in this Paper (section on theoretical analysis)
(*b*) the isotropic approximation
(*c*) the methods proposed by Schlosser & Vidal (1969) and by Juran & Schlosser (1979).

The estimation of Schlosser & Vidal turns out to be a lower bound solution derived from the yield design static approach performed on the homogenized wall. The stress field defined by Fig. 10

$$\sigma_{yy} = -\gamma y, \quad \sigma_{xx} = \sigma_{xy} = 0$$

where $-y$ represents the depth from the above free surface of the wall, is statically admissible with γ and is nowhere exceeding the macroscopic strength condition of reinforced earth, as long as $\gamma H \leqslant K_{\rm p}\sigma_0$.

Thence the following lower bound estimate is obtained for H^*

$$H^* = \frac{K^*\sigma_0}{\gamma} \geqslant K_{\rm p}\frac{\sigma_0}{\gamma} \tag{19}$$

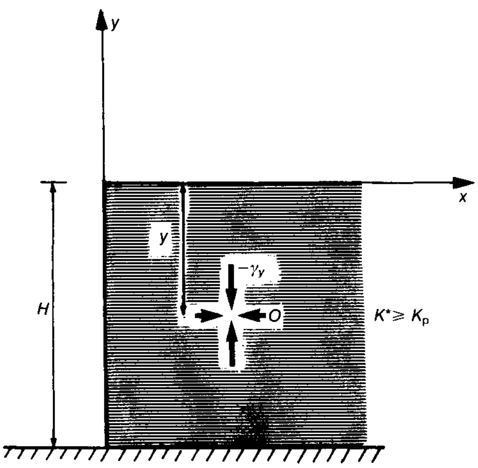

Fig. 10. Lower bound estimate for H^*, using yield design static approach

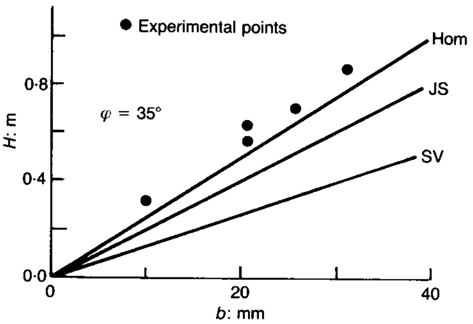

Fig. 11. Predicted against experimental (Ben Assila & El Amri, 1984) failure heights for model walls

However, the method proposed by Juran & Schlosser (1979) cannot conveniently be interpreted within the framework of the yield design theory. Therefore it is not possible to predict whether the result so obtained will provide an upper bound or a lower bound for the theoretical value H^*. It can only be observed from the comparison of the different curves plotted in Fig. 9, that the evaluation of H^* obtained that way lies between the upper bound kinematic estimate and the lower bound static one performed on the homogenized wall. Besides, this method would become unsuitable when applied to the design of other kinds of reinforced earth structures.

Comparison with experiments

Figure 11 shows a comparison between the theoretical failure heights estimated through the different methods mentioned and the experimental results obtained by Ben Assila & El Amri (1984) on reduced scale walls, using reinforcement strips of different width b, and thus different strength $R_{\rm t}$, since $R_{\rm t}$ is directly proportional to b. Friction angle φ of the backfill material has been taken equal to 35°, as suggested by those authors who carried out all tests on the same sand. The good performance of the homogenization method proposed is self evident, while the two other traditional methods (Schlosser & Vidal and Juran & Schlosser) significantly underestimate the actual failure heights.

As for the evaluation of H^* corresponding to the isotropic approximation described, the results obtained prove slightly higher than those proceeding from the homogenization method (the difference is less than 4% for $\varphi = 35°$) and appear therefore to be also in good agreement with experiments. Such a conclusion should only, however, be drawn for the specific case of vertical retaining walls, as it has been shown for instance that the same isotropic assumption could lead to

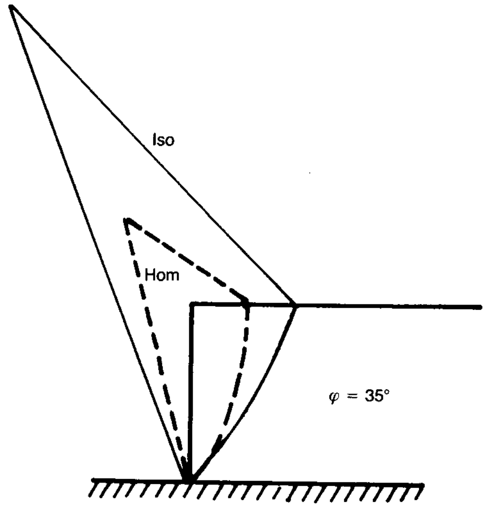

Fig. 12. Influence of anisotropy of reinforced earth on geometry of most critical failure mechanism

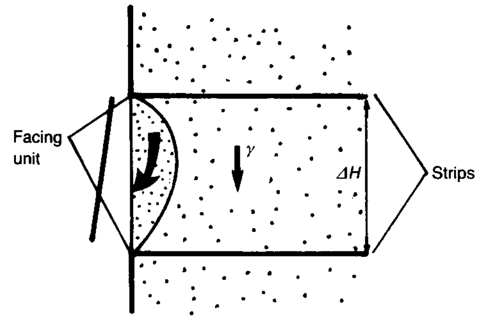

Fig. 13. Failure mechanism of backfill soil near facing of wall

highly overestimating the bearing capacity of a strip footing (de Buhan, Salençon & Siad, 1986). Besides, in spite of the relatively small gap existing between the two upper bound estimates, which can both be interpreted rigorously through the yield design theory and which make use of the same rigid block mechanisms, the corresponding optimal failure mechanisms prove to be quite different. The centre of rotation Ω for the optimum in the case of the anisotropic calculation is located far below the centre obtained from the isotropy approximation (Fig. 12).

Overall stabilizing effect of facing units

There is an apparent contradiction between the experimental data and the predictions given by the homogenization method. The proposed method should lead to an upper bound for the actual failure height, while in practice it slightly underpredicts it. If this discrepancy is considered significant, the following reason may be put forward as an explanation.

One of the characteristics of the homogenization theory, applied to reinforced earth structures, lies in the fact that boundary or edge effects cannot be taken into account. According to this theory, reinforced earth is modelled as a homogeneous material, and therefore no attention is paid to the heterogeneous nature of the constitutive soil, especially in the vicinity of the facing of the wall, where in practice the fitting of skin elements is necessary in order to prevent the soil from flowing away between the reinforcing layers. This building requirement can easily be understood from theoretical considerations, since there is always a possibility of a rigid block failure mechanism such as that sketched in Fig. 13, involving the soil between two successive layers. The work of the external forces W_e (weight of the soil) in that mechanism is strictly positive, while the corresponding resisting work W_r will be found equal to zero, as the soil is a cohesionless material. Consequently, the basic inequality ($W_e \leqslant W_r$) is no longer satisfied, which implies a local instability of the wall whatever its height, unless facing units made up either of concrete panels or steel would help to contain the backfill material.

To evaluate the contribution of the facing units to the overall stability of the wall, the experimental data derived from Ben Assila & El Amri (1984) can be considered. They can be interpolated by the linear formula: $H = 25b + 75$, where H and b are in millimetres. The stabilizing effect of the facing units can be estimated through the value of H when there is no reinforcement ($b = 0$), as in that case the theoretical failure height would be equal to zero.

If the experimental results are decreased by 75 mm, which can therefore reasonably be attributed to the facing units, the agreement between the observed data and the results calculated with the homogenization method becomes even better, as can be seen from Fig. 14.

Similar results can be obtained if the comparison is made with the experiments performed by Legeay (1978) on similar scale model walls. In both cases the observed failure line approximates a logspiral intersecting the above free surface at a distance of nearly 0·3 H from the facing, which can be favourably compared with the theoretical value of 0·33 H corresponding to the optimal failure mechanism obtained from the homogenization method.

CONCLUSION

The method proposed in this Paper, based on both the yield design theory and the concept of

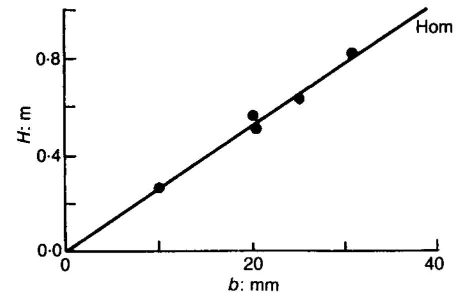

Fig. 14. Comparison of results predicted with homogenization method and experimental data depurated from facing unit effect

homogenization, results in a convenient approach for dealing with the stability analysis of reinforced earth constructions, provided the conditions that allow homogenization of the reinforced earth as an anisotropic material are satisfied. The method is still relevant when an additional failure condition relative to the interface between the soil and the reinforcing strip has to be taken into account. In that case, it can be shown (de Buhan & Siad, 1988) that the expression of the macroscopic strength criterion obtained in this Paper from the multilayered model needs to be modified only slightly.

Such a homogenization procedure provides a suitable theoretical framework for the collapse analysis of other kinds of reinforced soil structures: soil nailing, reinforcement by the inclusion of columns, micropiling and so on. The practical feasibility of the method relies in those cases on the possibility of numerical determination of the macroscopic strength criterion of the reinforcing soil, because analytical expressions as simple as those derived from the multilayered schematization are no longer met. New developments of an applied character are likely to appear in this field in coming years, because the yield design approach yields a handy computational tool not only for checking the stability of a given reinforced structure, but also for seeking its optimal design.

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APPENDIX 1. FUNDAMENTALS OF THE YIELD DESIGN THEORY (Salençon, 1983, 1984)

The scope of the yield design theory is to evaluate the failure loads of a structure, taking account of its geometry, loading conditions, and of knowledge of the strength criterion of the constituent material, being given no further information about its mechanical behaviour. The principle of the yield design approach is based on the following general statement.

Given any mechanical system or structure under loading conditions that depend on several parameters $\boldsymbol{Q} = (Q_1, \ldots Q_n)$, such as gravity forces or any other external forces applied to its boundary surface, and denoting by $G(x)$ the strength domain of the constitutive material at point x, i.e. the set of allowable stress tensors $\boldsymbol{\sigma}(x)$, it appears that a necessary condition for the structure to be safe is that there exists at least one stress field σ: satisfying all equilibrium equations and stress boundary conditions, and such that $\boldsymbol{\sigma}(x)$ belongs to $G(x)$ whatever the point x. This condition defines the domain of the loading $\boldsymbol{Q}$ for which the structure may be safe. The determination of this domain can be performed through two different approaches: static or kinematic.

Static or lower bound approach

The approach simply consists of exploiting the foregoing definition through the construction of statically admissible stress fields respecting the strength criterion everywhere. In the frequently encountered case of a unique positive loading parameter Q, this approach will provide a lower bound estimate for the failure load Q^*, defined as the maximum value of Q for which the structure may be safe.

Kinematic or upper bound approach

The kinematic approach is derived by dualizing the static approach through the principle of virtual work. Given any kinematically admissible virtual velocity field $\boldsymbol{v}$, and with the restriction of a single loading parameter, the following inequation can be written concerning Q^*

$$W_e(Q^*, \boldsymbol{v}) \leqslant W_r(\boldsymbol{v}) \tag{20}$$

where $W_e(Q^*, \boldsymbol{v})$ is the work of the external forces corresponding to Q^* calculated in the considered velocity field, while $W_r(\boldsymbol{v})$ is a positive functional (which might be named the maximum resisting work) defined as

$$W_r(\boldsymbol{v}) = \int_V \pi(\mathbf{d}(x))\, \mathrm{d}V + \int_\Sigma \pi(\boldsymbol{n}(x); [\![\boldsymbol{v}(x)]\!])\, \mathrm{d}\Sigma \tag{21}$$

with: $\mathbf{d}$ the strain-rate field derived from $\boldsymbol{v}$ by $d_{ij} = \frac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$; $[\![\boldsymbol{v}(x)]\!]$ the jump of the field $\boldsymbol{v}$ at point x when crossing a possible velocity discontinuity surface Σ following its normal $n(x)$, so that finally

$$\pi(\boldsymbol{d}(x)) = \sup\{-\sigma_{ij}(x)\, d_{ji}(x)^*,$$
$$\boldsymbol{\sigma}(x) \text{ belonging to } G(x)\}$$
$$\pi(\boldsymbol{n}(x); [\![\boldsymbol{v}(x)]\!]) = \sup\{-\sigma_{ij}(x) n_j(x) [\![v_i(x)]\!],$$
$$\boldsymbol{\sigma}(x) \text{ belonging to } G(x)\}$$

It follows from inequality (20) that the implementation of the kinematic approach leads to an upper bound estimate for Q^*.

Resuming the calculation performed where rigid body deformation modes are used, it is worth pointing out that this inequality is equivalent to

$$M_w \leqslant M_r$$

where M_w denotes the driving moment, i.e. the moment of the weight of the rotating block with respect to its centre of rotation, while M_r may be interpreted as the maximum resisting moment, developed in the opposite sense by the stress distribution (σ_n, τ_n) along the failure line on account of the strength capacities of reinforced earth. The mechanical significance of the kinematic approach is thus clearly brought out in this case, since the upper bound estimate of H^* derived from it corresponds to the smallest height of the embankment beyond which the overall moment equilibrium of the block can no longer be ensured without exceeding the strength criterion.

APPENDIX 2. MACROSCOPIC STRENGTH CONDITION OF REINFORCED EARTH

Combining equation (5) with strength conditions (2) and (4), in which σ_{xx} has been replaced by σ^s_{xx} or σ^r_{xx} respectively, the macroscopic strength condition may be expressed in the following equivalent form (see the section on formulation of a macroscopic strength criterion)

$$\begin{cases} (1-\eta)\sigma^s_- + \eta\sigma^r_- \leqslant \sigma_{xx} \leqslant (1-\eta)\sigma^s_+ + \eta\sigma^r_+ \\ \text{with} \quad |\sigma_{xy}| \leqslant \text{Min}\,\{C, \sigma_{yy}\tan\varphi\} \end{cases} \tag{22}$$

where $\sigma^s_+ \geqslant \sigma^s_-$ are the solutions of the following equation for σ^s

$$(\sigma^s - \sigma_{yy})^2 + 4\sigma^2_{xy} = (\sigma^s + \sigma_{yy})^2 \sin^2\varphi \tag{23}$$

and likewise $\sigma^r_+ \geqslant \sigma^r_-$ the solutions of the equation

$$(\sigma^r - \sigma_{yy})^2 + 4\sigma^2_{xy} = 4C^2 \tag{24}$$

Passing to the limit $\eta \to 0$, as $2\eta C = \sigma_0$ condition (22) reduces to

$$|\sigma_{xy}| \leqslant \sigma_{yy}\tan\varphi, \quad \text{since} \quad C \to +\infty \quad \text{as} \quad \eta \to 0$$

and

$$\sigma^s_- + \lim_{\eta\to 0}(\eta\sigma^r_-) \leqslant \sigma_{xx} \leqslant \sigma^s_+ + \lim_{\eta\to 0}(\eta\sigma^r_+)$$

Now from equation (24) there is obtained

$$\sigma^r_\pm = \sigma_{yy} \pm 2C[1 - \Sigma^2_{xy}/C^2]^{1/2}$$

whence

$$\lim_{\eta\to 0}(\eta\sigma^r_\pm) = \pm 2\eta C = \pm\sigma_0$$

The macroscopic strength condition can then be written

$$\begin{cases} \sigma^s_- - \sigma_0 \leqslant \sigma_{xx} \leqslant \sigma^s_+ + \sigma_0 \\ \text{with} \quad |\sigma_{xy}| \leqslant \sigma_{yy}\tan\varphi \end{cases}$$

or by introducing a varying parameter σ

$$\begin{cases} (\sigma_{xx} - \sigma - \sigma^s_+)(\sigma_{xx} - \sigma - \sigma^s_-) \leqslant 0 \\ |\sigma| \leqslant \sigma_0, \quad |\sigma_{xy}| \leqslant \sigma_{yy}\tan\varphi \end{cases}$$

that is finally

$$[(\sigma_{xx} - \sigma - \sigma_{yy})^2 + 4\sigma^2_{xy}]^{1/2} \leqslant (\sigma_{xx} - \sigma + \sigma_{yy})\sin\varphi$$

where $\qquad |\sigma| \leqslant \sigma_0.$

* Summation over repeated subscripts.

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