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BEARING CAPACITY OF A FOOTING ON A $\phi = 0$ SOIL WITH LINEARLY VARYING SHEAR STRENGTH

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Bearing capacity of a footing on a $\phi = 0$ soil with linearly varying shear strength

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This Note is concerned with the bearing capacity of a strip footing on a $\phi=0$ soil when the shear strength C increases linearly with depth from zero, i.e. according to the formula

$$C(z) = C_1 z/a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The width of the footing is B, the bearing capacity is Q_{ult} and the corresponding average pressure $P_{\text{ult}} = Q_{\text{ult}}/B$.

The determination of Q_{ult} is made within the framework of classical limit analysis, the material obeying Tresca's yield criterion with an associated flow rule.

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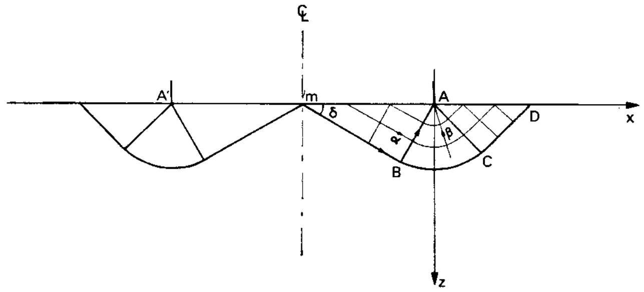

Fig. 1. Kinematic field: net of orthogonal zero extension lines

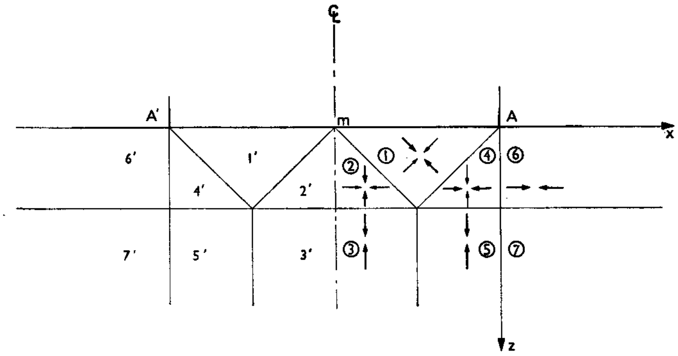

Fig. 2. Stress field

KINEMATIC APPROACH

Let us consider the class of symmetrical kinematic fields with no volume change, constructed by use of the orthogonal net of zero-extension lines α, β, shown in Fig. 1 and depending on parameter δ.

The footing is given a downwards vertical velocity U; the triangle mAB is undeformed and slips along mB with the velocity $U/\sin\delta$; under the footing the material slips with the velocity $U/\tan\delta$ along mA; the velocity jump across mBCD is tangent and equal to $U/\sin\delta$; ABC deforms plastically with no volume change and the corresponding velocity field is determined by means of Geiringer's equations

$$dv_\alpha - v_\beta \, d\theta = 0 \qquad \text{along } \alpha \text{ lines}$$

$$dv_\beta + v_\alpha \, d\theta = 0 \qquad \text{along } \beta \text{ lines}$$

thence

$$v_\alpha = U/\sin\delta, \; v_\beta = 0 \qquad \text{over ABC}$$

ACD is undeformed and slips along CD with velocity $U/\sin\delta$.

The rate of dissipation is obtained by integration along mBCD and over ABC, since the contribution along mA is zero, however rough or smooth the footing may be, as $C(\mathrm{o})=0$, hence

$$P(\delta) = C_1 U \frac{B^2}{2a}(1/2+\sqrt{2}\sin\delta+\sin^2\delta) \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

for the whole symmetrically deformed area, from which, applying the kinematic theorem of limit analysis,

$$P_{\text{ult}} \leqslant C_1 \frac{B}{2a}(1/2+\sqrt{2}\sin\delta+\sin^2\delta), \; \forall\delta\epsilon\left(0,\frac{\pi}{2}\right) \quad . \; . \; . \; . \; . \quad (3)$$

The minimum value of equation (2) is reached for $\delta = 0$ which corresponds to the limit of the kinematic fields considered. It then follows from equation (3) that

$$P_{\text{ult}} \leqslant \frac{1}{4} C_1 \frac{B}{a} \quad \ldots \ldots \ldots \ldots \quad (4)$$

STATIC APPROACH

Let us now consider the discontinuous symmetrical stress field in Fig. 2. Discontinuity lines are either horizontal, vertical, or inclined at $\pm\pi/4$ to the x-axis. The stresses in the different regions (tensions positive) are

$$\text{region 1: } \sigma_{zz} = C_1 x/a, \quad \sigma_{xx} = C_1 x/a, \quad \sigma_{zx} = -C_1 z/a$$

$$\text{region 2: } \sigma_{zz} = C_1 \frac{2x+B/2}{a}, \sigma_{xx} = C_1 \frac{2z-B/2}{a}, \sigma_{zx} = 0$$

$$\text{region 3: } \sigma_{zz} = C_1 \frac{2x+B/2}{a}, \sigma_{xx} = 0, \quad \sigma_{zx} = 0$$

$$\text{region 4: } \sigma_{zz} = C_1 2x/a, \quad \sigma_{xx} = -C_1 2z/a, \quad \sigma_{zx} = 0$$

$$\text{region 5: } \sigma_{zz} = C_1 2x/a, \quad \sigma_{xx} = 0, \quad \sigma_{zx} = 0$$

$$\text{region 6: } \sigma_{zz} = 0, \quad \sigma_{xx} = -C_1 2z/a, \quad \sigma_{zx} = 0$$

$$\text{region 7: } \sigma_{zz} = 0, \quad \sigma_{xx} = 0, \quad \sigma_{zx} = 0$$

σ_{yz} is zero everywhere, and σ_{yy} may be taken equal to σ_{xx}.

This stress field is statically and plastically admissible. The corresponding pressure under the footing along mA is

$$p(x) = -C_1 x/a \quad \ldots \ldots \ldots \ldots \quad (5)$$

and by use of the static theorem of limit analysis one gets

$$P_{\text{ult}} \geqslant \frac{1}{4} C_1 \frac{B}{a} \quad \ldots \ldots \ldots \ldots \quad (6)$$

CONCLUSION AND COMMENTS

Comparing equations (4) and (6)

$$P_{\text{ult}} = \frac{1}{4} C_1 \frac{B}{a} \quad \ldots \ldots \ldots \ldots \quad (7)$$

The exact value of P_{ult} is thus known. This result was indicated by Salençon *et al.* (1973) and by Davis and Booker (1973).

According to a general theorem (e.g. Salençon, 1974) under the assumption of existence of a limit equilibrium solution, the kinematic and static fields which lead to equation (7) should be associated. It is worth noting how this property is true here: in the static field of Fig. 2, region 1 is in a state of limit equilibrium where mA is an α line (as $\sigma_{zx} \leqslant 0$), point A in regions 4 and 6 is at limit equilibrium too; as $\delta = 0$ in Fig. 1 does not allow any movement, this case has to be considered as a limit. Therefore a complete solution to the problem under concern consists in the stress field of Fig. 2 which is associated with the limit of the kinematic field of Fig. 1 when $\delta \to 0$. (As far as is known, it is the first example of such a kind of limit equilibrium solution, which is obviously due to the condition $C(o) = 0$.)

In this complete solution the deformed region squeezes to mA. According to Hill's (1951) and Mandel's (1965) theorems, the distribution of stresses along mA is the exact one. It follows then from equation (5) that the pressure under the footing increases linearly with horizontal distance in the same ratio as the shear strength increases with depth. This confirms the conjecture made by Davis and Booker (1973), although the stress field proposed by these authors is not correct.

It was pointed out (Salençon *et al.*, 1973) that in the case of a weightless Coulomb material the resolution of the same problem as considered here leads to the formula

$$P_{\text{ult}} = \frac{1}{2} N_\gamma(\phi) \cot\text{an}\, \phi C_1 \frac{B}{a} \qquad (8)$$

Assuming the continuity of the solution of the problem with respect to ϕ as a physical likelihood, it follows from equation (7) that N_γ behaves as $\phi/2$ in the vicinity of $\phi=0$ both for rough and smooth footings.

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