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CAMBRIDGE UNIVERSITY ENGINEERING DEPARTMENT

PROCEEDINGS

OF THE

SYMPOSIUM ON THE ROLE OF PLASTICITY IN SOIL MECHANICS

CAMBRIDGE, 13-15 SEPTEMBER 1973

Edited by

Andrew C. Palmer

CAMBRIDGE

1973

LIMIT ANALYSIS BY FINITE-ELEMENT METHODS

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INTRODUCTION

Limit analysis - often in an implicit form - has been for long the principal application of the theory of Plasticity to Soil Mechanics. Through the so called statical and kinematical theorems, lower and upper bounds for failure loads can be obtained. As is well known, when dealing with plane strain or axi-symmetric problems statical or kinematical solutions are most often obtained using the method of characteristics - following the theory of slip line fields - or more or less roughly constructed by means of discontinuous fields, e.g. using slip circles. The latter method is the only one which is practically used for true three-dimensional problems.

In this paper the Authors intend to describe another approach by limit analysis, based on the kinematical theorem and using finite elements in a natural way; it is hoped that this method could be applied as well as the method of the slip surfaces and the results obtained either way could complete each other.

The study is made on the example of a plane strain problem for Tresca's or Mises's material. It appears that the knowledge of the general features of some exact solutions already available is of a great help when devising such a method : for example it shows the necessity of introducing discontinuities in the velocity fields. In the general case the problem is then reduced to convex programming of a non trivial kind.

The present attempt should be related to the developments of limit analysis for structures composed of beams (cf. Gavarini, 1972), or for plates (Hermann, 1967; Anderheggen, 1969, 1970; Anderheggen & Knöpfel, 1972; cf. Save, 1972).

SURVEY OF LIMIT THEOREMS

Let us consider a system made of a material satisfying Hill's maximum work principle (equivalent to : convexity of the yield criterion and normality of the flow rule) and let us assume that this system undergoes a loading process depending on n loading parameters Q_i. Q is the load vector.

The fundamental results of limit analysis theory can be stated as follows (Salençon, 1972) :

Any limit equilibrium solution for the system - (σ : stress field, d strain rate field) minimizes the functional

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$$I(\sigma, d) = P(d) - \underline{Q}(\sigma)\underline{\dot{q}}(d) \qquad (1)$$

among the set H of statically and plastically admissible (S.P.A.) stress fields, and the set H' of kinematically and plastically admissible (K.P.A.) strain rate fields.

P(d) is the rate of dissipation for the field d.

The load $\underline{Q}$ corresponding to such a solution is a limit load and $\underline{\dot{q}}$ is the associated strain rate for the system in consideration.

Then the statical theorem can be stated :

The yield locus - consisting of all the limit loads - of the system is the boundary of the convex K of all loads which can be equilibrated by S.P.A. stress fields.

From this, one derives the method for approaching the yield locus from the inside (fig. 1a), which leads to a problem of maximization in the case of radial loading (fig. 1b) :

$$\underline{Q}\ \lim = Q^{\circ}\mathrm{Max}\{\lambda(\sigma) \mid \lambda(\sigma)\underline{Q}^{\circ} = \underline{Q}(\sigma),\ \sigma \in H\} \qquad (2)$$

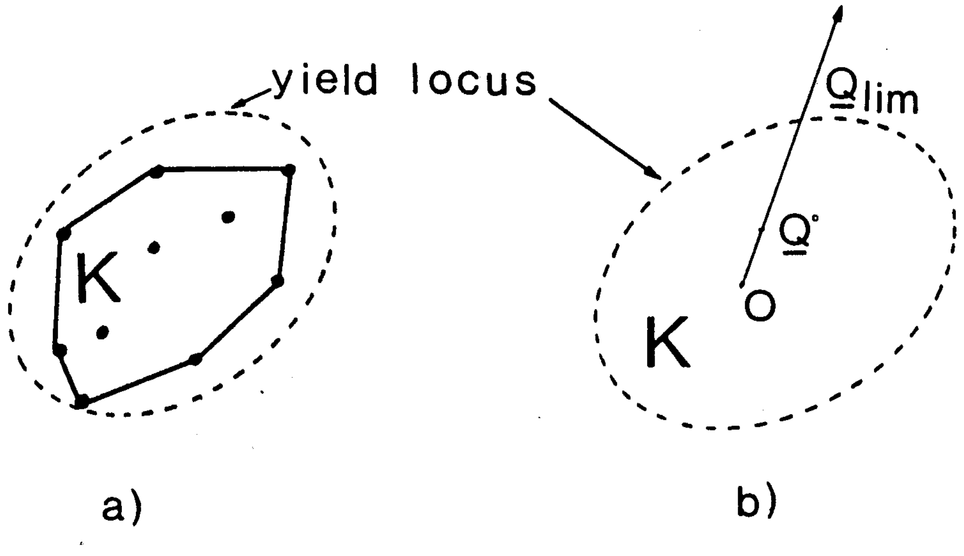

Figure 1. Statical method.

The kinematical theorem is also deduced from (1) :

In the space {Q} of load vectors, the set K is the intersection of the half-spaces defined for all K.P.A. fields d, by

$$P(d) - Q\ \dot{q}(d) > 0 \qquad (3)$$

This leads to the method for approaching the yield locus of the system from the outside (fig. 2a), which introduces a problem of minimization when one considers K.P.A. fields corresponding to the same $\dot{q}^0$ (fig. 2b) :

the plane

$$\underline{Q}\,\dot{\underline{q}}^0 - \mathrm{Min}\{P(d)\,|\,d \in H',\ \dot{\underline{q}}(d) = \dot{\underline{q}}^0\} = 0 \qquad (4)$$

is tangent to the yield locus.

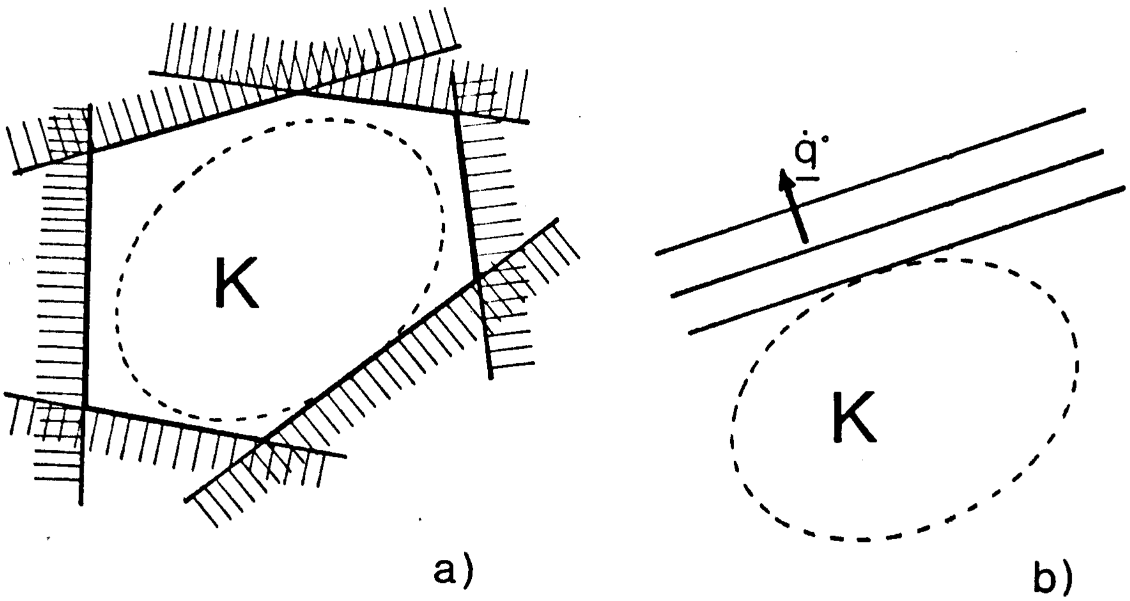

Figure 2. Kinematical method.

OUTLINE of PROPOSED METHOD

It is natural to try constructing stress fields or strain rate fields for the above mentionned theorems, by means of finite elements. From this point of view the kinematical theorem seems more convenient as it allows us to draw the mesh in a priori restricted regions, the rest of the system being given a solid motion defined by the boundary conditions. This possibility proves very useful when dealing with systems of great dimensions or even unlimited in some directions.

The application of such a method will be studied on a plane strain problem for a system made of Tresca's or Mises's material the limit shear stress of which is noted k.

It is to be reminded that for such a material, P.A. strain rate fields are all the fields with no local volume change: $\mathrm{tr}\, d = \mathrm{div}\, \vec{v} = 0$

Besides, generally speaking, it is known that the kinematical theorem allows us to use fields d derived from discontinuous P.A. velocity fields $\vec{v}$; in the case of Tresca's (or Mises's) material the jump $[\vec{v}]$ has to be tangent to the surface of discontinuity the rate of dissipation for Mises's material is given by

$$P(d) = \int k\sqrt{2d_{ij}d_{ji}}\, d\Omega + \int k|\,[\vec{v}]\,|\,d\omega \qquad (5)$$

where the first integral is taken over all strained regions, and the latter concerns the jump surfaces. Equation (5) is also valid for Tresca's material in plane strain, which will be considered in the following sections.

AN EXAMPLE

The problem which is chosen as an example and as a test for the method, though it may seem rather academic, is of importance for soil mechanists as it is connected with the bearing capacity of a foundation near a slope, and with the passive pressure on a retaining wall for certain conditions of motion.

It concerns a smooth rigid plate OA (width 2a) acting on an infinite rectangular wedge constituted of weightless Tresca's material. Separation is allowed between plate and soil. The flow rule of this contact is associated and allows us to use classical limit theorems.

Introducing the resultant force N and resultant moment M with respect to the middle point I of the plate, one can use the non dimensional load- and strain-parameters :

$$Q_1 = N/ak,\ Q_2 = M/a^2k$$
$$\dot{q}_1 = -v_y(I)/|\omega|a,\ \dot{q}_2 = \omega/|\omega| = \mathrm{sgn}\ \omega \tag{6}$$

where ω is the angular velocity of the plate; the lengths being compared to a, and the rate of dissipation to $|\omega|a^2k$.

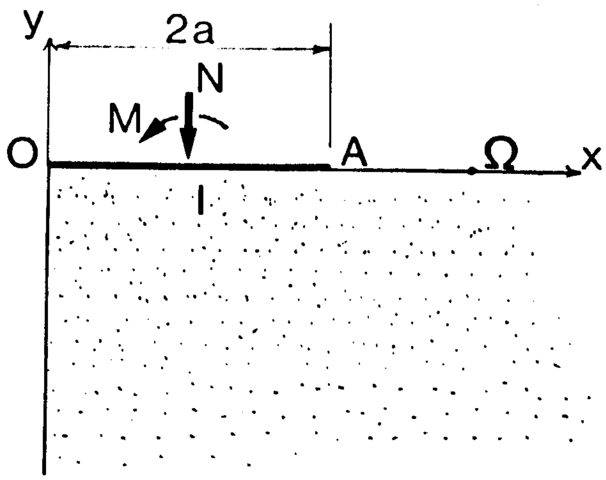

Figure 3

Results concerning the yield locus of that system (Salençon,1973), obtained using more or less classical methods, are shown on fig. 4.

① is the exact yield locus corresponding to $\omega > 0$, wherever the center of rotation Ω may be :

$$M/a^2k > 0,\ M/a^2k = N/ak(1-N/4ak) \tag{7}$$

② is the exact yield locus for $\omega < 0$ and $x(\Omega)/a > 1.584$:

$$M/a^2k < 0,\ M/a^2k = -N/ak[1-N/(2\pi+4)ak]$$
$$N/ak < 2.136 \tag{8}$$

③ is an approximation of the yield locus from the outside. It is the continuation of parabola (8) for $2.136 < N/ak < 4$

④ is an approximation of the yield locus from the outside : a vertical tangent at the point ($N/ak = 4$, $M/a^2k = 0$).

⑤ is a numerically obtained approximation from the inside.

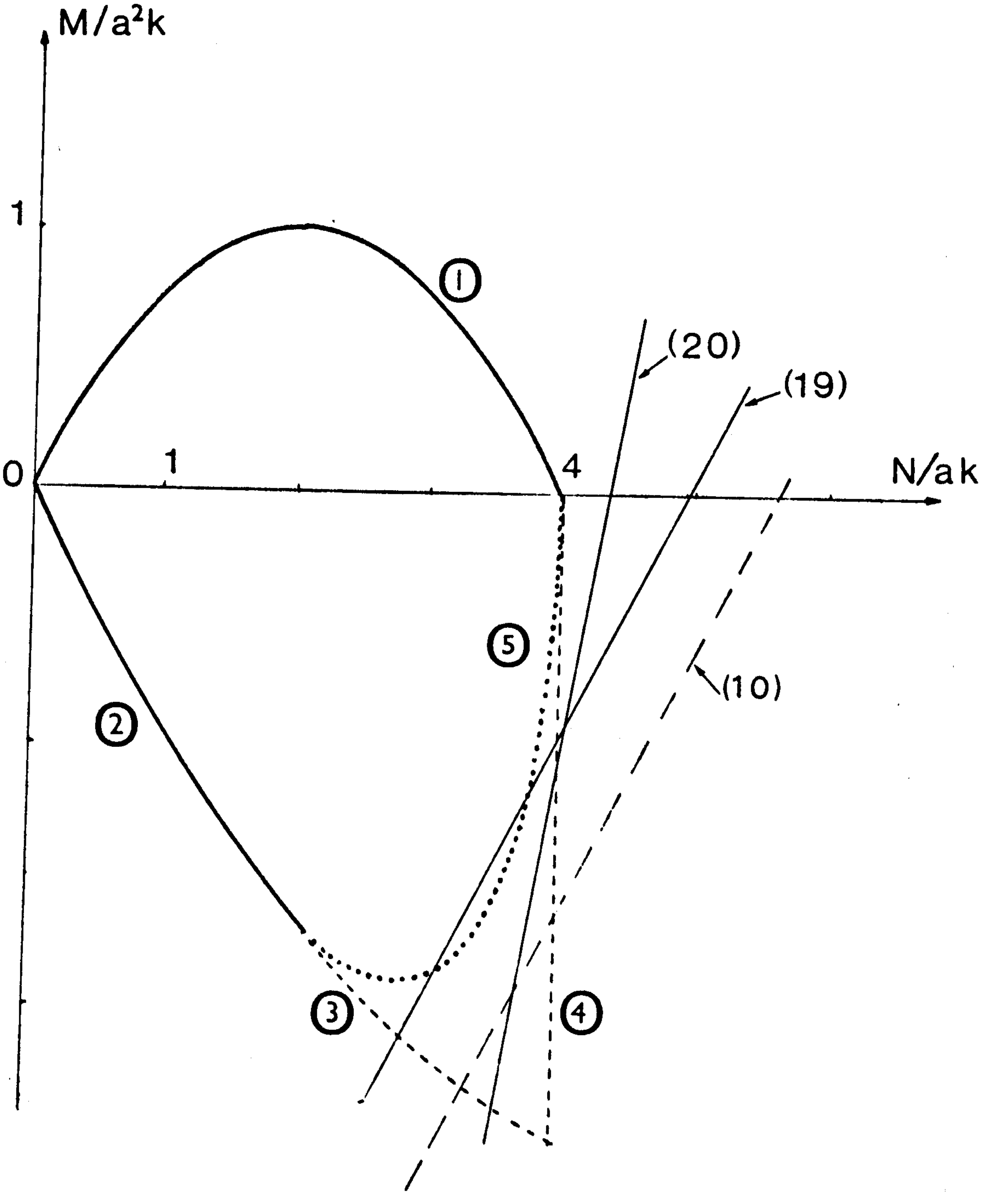

Figure 4

Through the use of slip circles, better approximations of the yield locus from the outside are obtained for :

$$\omega < 0, \; x(\Omega)/a < 1.584 \qquad (9)$$

For instance, for $x(\Omega)/a = 0$, which corresponds to $\dot{q}_1^0 = 1$, $\dot{q}_2^0 = -1$, the best slip circle is classically known and yields the equation (10) for plane (4) which is drawn on fig. 4 :

$$N/ak - M/a^2k - 5.56 = 0 \qquad (10)$$

With the finite element method for limit analysis proposed in the paper we shall study the range (9).

APPLICATION OF THE METHOD

A mesh of t triangular elements is drawn over the cross section of the wedge as shown on fig. 5. The deformable zone is reduced to a square of sufficient width. Using that mesh, various kind of K.P.A. fields were constructed which are described below, the velocity being equal to zero outside the meshed square.

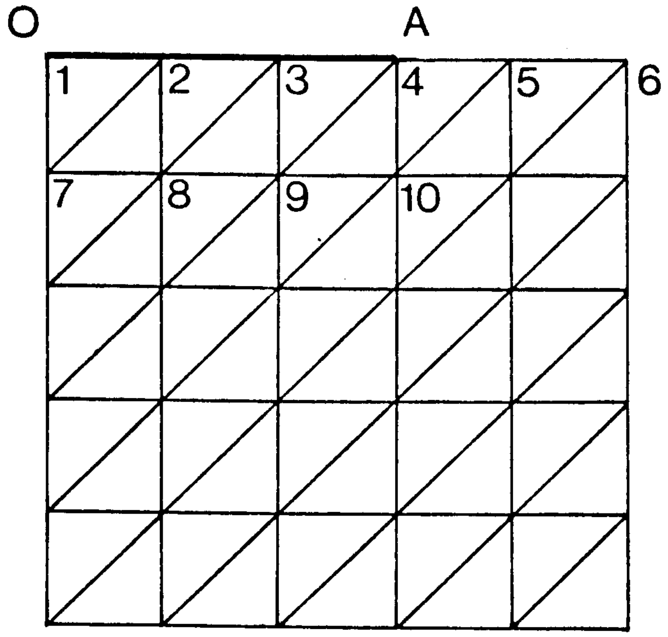

Figure 5

1st Model : Linear variation of $\dot{v}$. Div $\vec{v} = 0$ on each triangular element

The velocity $\vec{v}$ varies linearly in each triangle, and is continuous on the whole wedge. Such a field is described by the 2n-component vector u built with the values of the velocity components at the vertices of the mesh. The field will be strictly P.A. if div $\vec{v} = 0$ is true everywhere. The strain rate being constant on each element, this yields t linear conditions.

Boundary conditions prescribe the values of some components of u at some of the vertices along the boundary of the mesh :

$$Du = u^d$$

where D is a projection matrix, and u^d is the vector of the prescribed component values. In fact, to account for the possibility of separation along OA, Boundary conditions should be divided into two parts $D_1u = u_1^d$ and $D_2u \leq u_2^d$, but this does not affect the reasoning anyway.

The rate of dissipation for each element is the square root of a quadratic form; thence :

$$P(d) = \int k\sqrt{2d_{ij}d_{ji}}\, d\Omega + 0 = \sum_{i=1}^{t} \sqrt{u^T A_i u} = P_1(u) \qquad (11)$$

The number of degrees of freedom for minimizing P_1 is very small in this model because, as it is known (Anderheggen & Knöpfel, 1972), $t/n \rightarrow 2$ when $n \rightarrow \infty$: therefore this number will be of order $\sqrt{n}$ when n increases. Thus this model is of little practical use but leads to a variant, with a reduced number of conditions.

2nd Model : linear variation of $\vec{v}$; Div $\vec{v} = 0$ in the mean sense on each square element.

The only alteration to the 1st model concerns the condition div $\vec{v} = 0$.

Div $\vec{v} = 0$ in the mean sense on a square element made of two triangular elements of the mesh, such as 2389 , if the integral of div $\vec{v}$ over this square element equals zero.

Considering fields which satisfy only this reduced condition on each square element, we get for u the condition

$$Bu = 0$$

where B is only a$(t/2)\times 2n$ matrix.

These fields are no longer strictly P.A., but they tend to become P.A. when $n \rightarrow \infty$.

The problem to be solved is then :

$$\text{Inf}\{P_1(u) \mid Bu = 0,\ Du = u^d\} \qquad (12)$$

which was transformed into the saddle-point problem (13) :

$$\underset{Du=u^d}{\text{Inf}}\ \underset{\mu}{\sup}\{P_1(u) + \mu^T Bu\} \qquad (13)$$

where μ is a $t/2$ vector. Problem (13) is solved using iterative methods (descent method with respect to u, gradient method with respect to μ).

The dualization process used to go from (12) to (13) can be given a mechanical interpretation which helps to understand the meaning of solutions so obtained.

Let us consider a "standard" material with yield criterion defined by

$$1/2\ \text{tr}\ s^2 \leq k^2,\ 3\ell^- \leq \text{tr}\ \sigma \leq 3\ell^+$$

i.e. Mises's Criterion with restricted value of the mean stress tr $\sigma/3$. Any strain rate field is P.A. for that material and the rate of dissipation is

$$P(d) = \int [k\sqrt{2d_{ij}d_{ji}} + \operatorname{Sup}\{\ell \ \mathrm{tr}\,\sigma \mid \ell^- < \ell < \ell^+\}d\Omega \qquad (14)$$

For a system made of such a material, the kinematical theorem will lead to the minimization of (14) among all K.A. velocity fields, and after discretization and taking the mean value of div $\vec{v}$, to :

$$\operatorname*{Inf}_{Du=u^d} \{P_1(u) + \operatorname*{Sup}_{\ell^-<\ell<\ell^+} \{\ell^T Bu\}\}$$

where ℓ is a t/2 component vector.

Then in order to recover Mises's material one only has to make $-\ell^-$ and ℓ^+ tend to infinity. This brings to problem (13).

The method was tested on the example. The width of the meshed square was 4a.

In a first case, for $x(\Omega)/a = 2$, $\omega > 0$, $(\dot{q}_1^0 = 1, \dot{q}_2^0 = 1)$ the exact solution is already known, the minimum value of P being 4. The method gives the value $P_1 = 4.1$ after 30 iterations, and the corresponding field is very similar to the exact one.

In a second case for $x(\Omega)/a = 0$, $\omega < 0$, using a 121 nodes mesh, 70 iterations lead to the value $P_1 = 7.312$, whereas the slip-circle method yields the value $P = 5.56$ as already said.

From these two trials one can see that : for the 1st case, as the exact velocity field is linear and continuous, the rigourous method of the 1st model cannot miss it. Since this solution is reached rather quickly by the 2nd model, we can have good confidence in the two approximations concerning div $\vec{v} = 0$ made in this model; for the 2nd case, the experience derived from the knowledge of exact solutions for similar problems makes us believe that a discontinuity line should exist starting from A. The model proposed here is evidently badly suited to describe such a solution.

<u>3rd Model : Discontinuous velocity field. Constant velocity on each triangular element.</u>

The velocity is constant on each triangular element, discontinuity of the velocity may occur between adjacent elements. Such a field is described by the 2t-vector $\hat{u}$ built with the components of the velocity on each element. In order to get a P.A. field the velocity jumps $[\vec{v}]$ must be tangent to the sides of the triangles: this gives the condition

$$\hat{B}\hat{u} = 0$$

where $\hat{B}$ is a s×2t matrix.

Boundary conditions are satisfied in the mean sense :

$$\hat{D}\hat{u} = \hat{u}^d$$

The rate of dissipation is

$$P(d) = \int k|[\vec{v}]|d\omega = \sum_{i,j} c_{ij}|\hat{u}_i-\hat{u}_j| = P_2(\hat{u}) \quad (16)$$

the numerical problem is then :

$$\mathrm{Inf}\{P_2(\hat{u}) \mid \hat{B}\hat{u} = 0,\ \hat{D}\hat{u} = \hat{u}^d\} \quad (17)$$

which can be reduced to a linear program in the canonical form.

This model was tested in the case $x(\Omega)/a = 0$, $\omega < 0$:

for a width of the meshed square of 3a, and a mesh of 72 triangles, one gets the minimum $P_2 = 6.75$

for a width of 2.5a and 200 triangles, one gets $P_2 = 6.8125$.

for a width of 2.22a and 800 triangles, after 3220 iterations, one gets the value $P_2 = 7.9205$.

All these results are worse than the value obtained by slip circle method. Moreover, we see that the minimum of P_2 increases as the number of elements increases. This is no paradox and proceeds from boundary conditions being better satisfied as the mesh is refined.

This model is not adequate; it allows us minimizing over a wide range of discontinuity lines, but deformation of the elements themselves is impossible. Another model was imagined combining features of the two preceeding ones.

4th Model : linear variation of $\vec{v}$. Div $\vec{v} = 0$ in the mean sense discontinuous velocity.

A curve Γ made of horizontal or vertical straight boundaries of elements is drawn through the mesh. The velocity varies linearly on each element and can admit discontinuity on Γ; the velocity jump $[\vec{v}]$ must be tangential to Γ on account of div $\vec{v} = 0$. It is worth noting that the velocity jump is continuous and piecewise linear along Γ; it is equal to zero at any vertex of Γ. Such a field is described by a vector u built with the velocity components at the vertices of the mesh, the nodes situated on Γ appearing twice for the vertical or horizontal components.

The rate of dissipation is given by :

$$P(d) = \sum_i \sqrt{u^T A_i u} + \sum_{j,k} f(|u_j-u_k|) = P_3(u) \quad (18)$$

Boundary conditions, and div $\vec{v} = 0$ in the mean sense are expressed as in the 2nd model, and the numerical method used to solve the problem is the same.

The minimization of P_3 for a given line Γ can be combined with the optimization of the shape of Γ.

For the present problem, in the different cases which were tested, it is natural to have Γ starting from A. Various likely shapes were tried. The width of the meshed square was 4a.

For $x(\Omega)/a = 0$, $\omega < 0$, with a mesh of 200 triangles, the value 4.93 was, obtained for P_3 after 230 iterations. This gives the plane (19) drawn on fig. 4 :

$$N/ak - M/a^2k - 4.93 = 0 \qquad (19)$$

For $x(\Omega)/a = -2$, $\omega < 0$, with a mesh of 72 triangles, the value 13.03 was obtained. The shape of the discontinuity-line and the velocity field yielding this result are shown on fig. 6. It corresponds to the plane (20) on fig. 4 ($\dot{q}_1^0 = 3$, $\dot{q}_2^0 = -1$) :

$$3N/ak - M/a^2k - 13.03 = 0 \qquad (20)$$

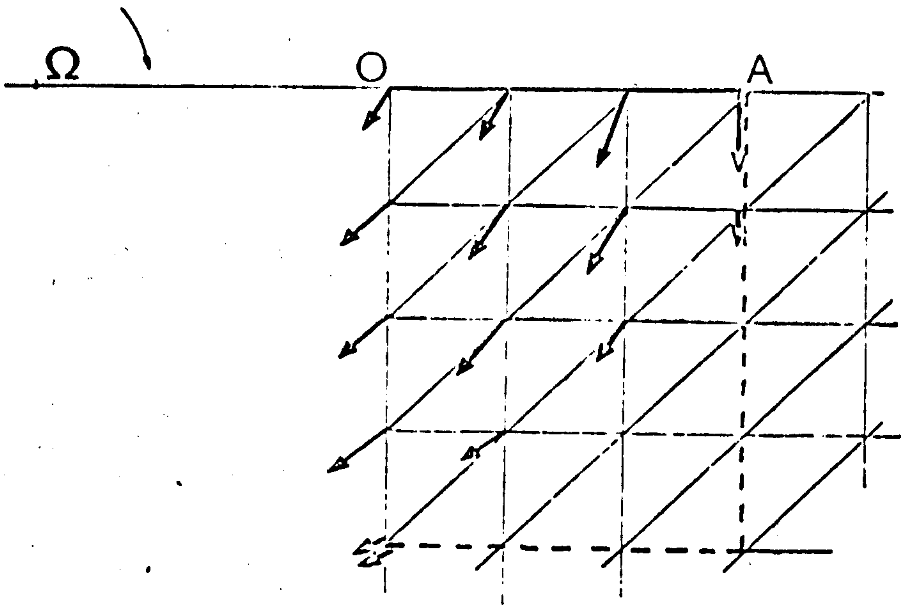

Figure 6

In the first case the result is better than the value 5.56 obtained using a slip-circle, but it is too good now as it appears that the plane (19) intersects the statical approximation ⑤ ! This could seem surprising since the method is based on the kinematical theorem, but it should be remembered that the fields considered in the model are only approximately P.A. This shows the limitations of the method.

In the second case the plane (20) proves to be very good. Of course it should not be taken as an accurate approximation from the outside, but as an approximation with some uncertainty.

It should be emphasized that we are looking here for an accuracy that would not be necessary in usual cases, since the already available results for the example chosen for testing the method could be considered as sufficient for any practical application.

COMMENTS

Four models have been progressively studied in this paper. The last one appears the most adequate, as it makes use of non zero strain rate fields derived from discontinuous K.P.A. velocity fields. This is in accordance with the fact that for some exacts solutions available, lines of discontinuity do exist but do not concentrate all the deformation of the system.

It is worth noting that the conditions defining K.P.A. fields are dealt with differently in the four models: only the first model does rigourously fulfil all conditions but it is of little practical use; in the 3rd model the fields are rigourously P.A. but satisfy boundary conditions in the means sense only; with the 4th model, boundary conditions are exactly satisfied but the fields are P.A. only in the limit since the condition div $\vec{v} = 0$ is taken in the mean sense and dualized.

This does not allow us to use the kinematical theorem rigorously: we cannot be sure that a so obtained minimum of P_i is superior to the true minimum of P, i.e. we cannot be sure to have an approximation of the yield locus from the outside. The mechanist should feel if the approximation is sufficient, as he always has to do with numerical method . Nevertheless, the results obtained so far using mean values and dualization give confidence in that approach which appears the most efficient one available at the moment.

Improvements in the 4th model could be obtained by diversifying the possible shapes for Γ as it has been seen that the results are highly dependent on it: also the numerical procedure could certainly become more efficient, as for the present time calculations are still rather long.

As a conclusion, the Authors believe that finite element method can lead to new developments in limit analysis, because of its ability to use elaborate fields in a systematical way. There are a lot of problems of practical importance and especially problems connected with soil mechanics, for which one would greatly appreciate any new result sufficiently accurate for practical use.

The method is indeed based on the classical limit theorems which are not practically valid when dealing with soils obeying Coulomb's criterion. In that case we should have to refer to the extended limit theorems given by Radenkovic (1961), Palmer (1966).

The extension of the method to three-dimensional problems does not rise any theoretical difficulty and would be of the highest interest, but the numerical difficulties involved are too big, for the present.

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ORAL DISCUSSION OF THE PAPER BY J. SALENÇON

E.H. Davis

You're dealing entirely with $\phi = 0$ material in all the work you're doing, so in that case your statement that the jump in velocity is tangential is alright. There is then a further question: in your lower bound solution I imagine all your difficult regions are those in the lower part of your main diagram. The lower bound solutions surely involve stress discontinuity, did they not?

J. Salençon

Yes, you must remember, it was using Bishop's complete solution for the punch indentation problem, and it involves stress discontinuity.