From phenomena of synchronization to boundary synchronization for hyperbolic systems

Li Tatsien

Fudan University
Shanghai, China

November 12, 2018

At the IAS Conference Celebrating France-Hong Kong Scientific Cooperation in Honour of Professor Philippe G.Ciarlet’s 80th Birthday
1. In as early as 1982 Philippe made his first visit to China, and afterwards he has been to China many times to visit different universities and research institutions.
2. He was nominated Honorary Professor of Fudan University in 1994 and of Xi’an Jiaotong University in 2006.
3. He is one of the founding members of ISFMA (Institut Sino-Français de Mathématiques Appliquées), created in February, 1998.
4. His six monographs and textbooks, including his recent book "Linear and Nonlinear Functional Analysis and Applications", have been translated into Chinese with great influence.
6. He gained the International Scientific and Technological Cooperation Award conferred by Shanghai Municipality in 2006.
7. He was elected a foreign member of the Chinese Academy of Sciences in 2009.
8. Due to all the above, he is a reliable friend of Chinese mathematicians and a great ambassador who facilitates exchange and cooperation between China and France in applied mathematics.

A conference dedicated to him on the occasion of his 80th birthday was organized on May of this year in Shanghai.
1. Introduction
Synchronization is a widespread natural phenomenon. Thousands of fireflies may twinkle at the same time; audiences in the theater can applaud with a rhythmic beat; pacemaker cells of the heart function simultaneously; and field crickets give out a unanimous cry. All these are phenomena of synchronization.
In principle, synchronization happens when different individuals possess likeness in nature, that is, they conform essentially to the same governing equation, and meanwhile, the individuals should bear a certain coupled relation.
The phenomenon of synchronization was first observed by Huygens in 1665.

**Figure:** Huygens and his pendulum experiment.
The research on synchronization from a mathematical point of view dates back to Wiener in the 1950s.

**Figure:** Norbert Wiener
The previous studies focused on systems described by ODEs, such as

$$\frac{dX_i}{dt} = f(X_i, t) + \sum_{j=1}^{N} A_{ij} X_j, \quad i = 1, \ldots, N, \quad (1)$$

where $X_i \ (i = 1, \ldots, N)$ are $N$-dimensional state vectors, $A_{ij} \ (i, j = 1, \ldots, N)$ are $n \times n$ coupling matrices, and $f(X, t)$ is an $n$-dimensional vector function independent of $i = 1, \ldots, N$. 
The right-hand side of (1) shows that every $X_i$ ($i = 1, \cdots, N$) possesses two basic features, that is, satisfying a fundamental governing equation and bearing a coupled relation among one another.
If for any given initial data

\[ t = 0 : \quad X_i = X^{(0)}_i \quad (i = 1, \cdots, N), \quad (2) \]

the solution \( X = (X_1, \cdots, X_N)^T = X(t) \) to the system satisfies

\[ X_i(t) - X_j(t) \to 0 \quad (i, j = 1, \cdots, N) \quad \text{as} \quad t \to +\infty, \quad (3) \]

namely, all the states \( X_i(t) \quad (i = 1, \cdots, N) \) tend to coincide with each other as \( t \to +\infty \), then we say that the system possesses the \textbf{synchronization in the consensus sense}. 

Li Tatsien (Fudan University)
November 12, 2018 17 / 146
Or, in particular, if the solution $X = X(t)$ satisfies

$$X_i(t) \to a \quad (i = 1, \cdots, N) \quad \text{as } t \to +\infty,$$

where $a$ is a constant state which is a priori unknown, then we say that the system possesses the synchronization in the pinning sense.

Obviously, the synchronization in the pinning sense implies that in the consensus sense.
These kinds of synchronizations are all called the asymptotic synchronization which should be realized on the infinite time interval $[0, +\infty)$. 
What we have been doing in the recent years with RAO Bopeng is to extend, in both concept and method, the universal phenomena of synchronization from finite dimensional dynamical systems of ordinary differential equations to infinite dimensional dynamical systems of partial differential equations (PDEs).
This should be the first attempt in this regard, and the earliest relevant paper was published in 2013 in a special issue of Chinese Annals of Mathematics (Series B) in honor of the scientific heritage of J.-L. Lions, the results of which were announced in 2012 in CRAS.
We will consider the following coupled system of wave equations with Dirichlet boundary conditions:

\[
\begin{cases}
U'' - \Delta U + AU = 0 & \text{in } (0, +\infty) \times \Omega, \\
U = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\
U = DH & \text{on } (0, +\infty) \times \Gamma_1
\end{cases}
\]

with the initial data

\[ t = 0 : U = \hat{U}_0, \ U' = \hat{U}_1 \quad \text{in } \Omega, \]

where \( \Omega \) is a bounded domain with smooth boundary \( \Gamma = \Gamma_0 \cup \Gamma_1 \).
\[
\begin{aligned}
U'' - \Delta U + AU &= 0 \quad \text{in} \ (0, +\infty) \times \Omega, \\
U &= 0 \quad \text{on} \ (0, +\infty) \times \Gamma_0, \\
U &= DH \quad \text{on} \ (0, +\infty) \times \Gamma_1
\end{aligned}
\]

\(U = (u^{(1)}, \ldots, u^{(N)})^T\) is the state variable, \(A\) is an \(N \times N\) coupling matrix with constant elements, \(D\), called the boundary control matrix, is an \(N \times M\) full column-rank matrix \((M \leq N)\) with constant elements, \(H = (h^{(1)}, \ldots, h^{(M)})^T\) stands for the boundary control.
A boundary control matrix $D$ is added to the boundary condition on $\Gamma_1$. This approach is more flexible: we will see in what follows that the introduction of $D$ enables us to simplify the statement and the discussion to a great extent.
For systems governed by PDEs, we can similarly consider the asymptotic synchronization on an infinite time interval as in the case of systems governed by ODEs, namely, we may ask the following questions: under what conditions do the system states with any given initial data possess the asymptotic synchronization in the consensus sense:

\[ u^{(i)}(t, \cdot) - u^{(j)}(t, \cdot) \to 0 \quad (i, j = 1, \ldots, N) \]  

(7)
as \( t \to +\infty \), or, in particular, if the system states with any given initial data possess the asymptotic synchronization in the pinning sense:

\[ u^{(i)}(t, \cdot) \to u(t, \cdot) \quad (i = 1, \cdots, N) \text{ as } t \to +\infty. \]  

(8)

where \( u = u(t, \cdot) \) is called the asymptotically synchronizable state, which is a priori unknown?
If the answer of this question is positive, these conclusions should be realized spontaneously on an infinite time interval $[0, +\infty)$, and is a naturally developed result decided by the nature of the system itself.
But for systems governed by partial differential equations, as there are boundary conditions, another possibility exists, i.e., to give artificial intervention to the evolution of state variables through appropriate boundary controls, which combines synchronization with control and introduces the study of synchronization to the field of control. This is also a new perspective to the investigation of synchronization for systems of partial differential equations.
The boundary control comes from the boundary condition $U = DH$ on $\Gamma_1$. The elements in $H$ are adjustable boundary controls, the number of which is $M(\leq N)$. To put the boundary control matrix $D$ before $H$ will provide many possibilities of combining boundary controls.
Precisely due to the artificial intervention of control, we can make a higher demand, i.e., to meet the requirement of synchronization within a limited time, instead of waiting until \( t \to +\infty \).
The corresponding question is whether there is a suitably large $T > 0$, such that for any given initial data $(\widehat{U}_0, \widehat{U}_1)$, through proper boundary controls with compact support in $[0, T]$ (that is, to exert the boundary control at the time interval $[0, T]$, and abandon the control from the time $t = T$), the solution $U = U(t, x)$ to the corresponding problem (5)-(6) satisfies, as $t \geq T$,

$$u^{(1)}(t, \cdot) \equiv u^{(2)}(t, \cdot) \equiv \cdots \equiv u^{(N)}(t, \cdot) := u(t, \cdot),$$

(9)

that is, all state variables tend to be the same since the time $t = T$, while $u = u(t, x)$ is called the corresponding exactly synchronizable state which is unknown beforehand.
If the above is satisfied, we say that the system possesses the **exact boundary synchronization**. Here, ”exact” means that the synchronization of state variables is exact without error, and the so-called ”boundary” indicates the means or method of control, i.e., to realize the synchronization through boundary controls.
In the above definition of synchronization, through boundary controls on the time interval \([0, T]\), we not only demand synchronization at the moment \(t = T\), but also require synchronization to continue when \(t \geq T\), i.e., after all boundary controls are eliminated. This kind of synchronization is not a short-lived one, but exists once and for all, as is needed in applications.
We always assume that the initial value
\((\hat{U}_0, \hat{U}_1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N\), and the
solution to problem (5)-(6) belongs to the
corresponding function space, which will not be
specified one by one here.
The synchronization considered in the framework of classical solutions in the one-space-dimensional case have been also considered.
2. Exact boundary null controllability
The exact boundary synchronization on a finite time interval is closely related to the exact boundary null controllability.
If there exists $T > 0$, such that for any given initial data $(\hat{U}_0, \hat{U}_1)$, through boundary controls with compact support in $[0, T]$, the solution $U = U(t, x)$ to the corresponding problem (5)-(6) satisfies, as $t \geq T$,

$$u^{(1)}(t, x) \equiv u^{(2)}(t, x) \equiv \cdots \equiv u^{(N)}(t, x) \equiv 0, \quad (10)$$

then we say that the system possesses the so-called exact boundary null controllability in control theory. This is of course a very special case of the above mentioned exact boundary synchronization.
For a single wave equation, the **exact boundary null controllability** can be proved by the HUM method proposed by J.-L. Lions. For a coupled system of wave equations, since for the purpose of studying the synchronization, the coupling matrix $A$ should be an arbitrarily given matrix, the proof of the exact boundary null controllability can not be simply reduced to the case of a single wave equation, however, using a compact perturbation result, it is possible to establish a corresponding observability inequality for the adjoint system, and then the HUM method can be still applied.
(1) Assume that \( M = N \) and let the domain \( \Omega \) satisfy the usual multiplier geometrical condition: there exists \( x_0 \in \mathbb{R}^n \), such that, setting \( m = x - x_0 \), we have

\[
(m, \nu) > 0, \forall \ x \in \Gamma_1, \quad (m, \nu) \leq 0, \forall \ x \in \Gamma_0, \quad (11)
\]

where \( \nu \) is the unit outward normal vector, and \((\cdot, \cdot)\) denotes the inner product in \( \mathbb{R}^n \).

Then through \( N \) boundary controls, when \( T > 0 \) is suitably large, the exact boundary null controllability can surely be realized for all

\[
(\hat{U}_0, \hat{U}_1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N.
\]
(2) Assume that $M < N$, that is, if the number of boundary controls is fewer than $N$, then no matter how large $T > 0$ is, the exact boundary null controllability cannot be achieved for all

\[(\hat{U}_0, \hat{U}_1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N.\]
In the case of partial lack of boundary controls, which kind of controllability in a weaker sense can be realized by means of fewer boundary controls? It is a significant problem from both theoretical and practical points of view, and can be discussed in many different cases as will be shown in the sequel.
3. Exact boundary synchronization
Really meaningful synchronization should exclude the trivial situation of null controllability, and thus we get the following results:
Assume that the system under consideration is exactly synchronizable, but not exactly null controllable, that is to say, assume that the system is exactly synchronizable with \( \text{rank}(D) < N \). Then the coupling matrix \( A = (a_{ij}) \) should satisfy the following condition of compatibility:

\[
\sum_{p=1}^{N} a_{kp} = a \quad (k = 1, \cdots, N),
\]

where \( a \) is a constant independent of \( k = 1, \cdots, N \), namely, the sum of all elements in every row of \( A \) is the same (the row-sum condition).
Let

\[ e_1 = (1, \ldots, 1)^T. \]

The condition of compatibility (12) is equivalent to that \( e_1 \) is an eigenvector of \( A \), corresponding to the eigenvalue \( a \).
Let

\[ C_1 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ \vdots & \vdots \\ 1 & -1 \end{pmatrix} \]  \hspace{1cm} (N-1) \times N \tag{13} \]

be the corresponding synchronization matrix. $C_1$ is a full row-rank matrix. Obviously, the synchronization requirement (9) can be written as

\[ t \geq T : \quad C_1 U(t, x) \equiv 0. \tag{14} \]
Moreover, $\text{Ker}(C_1) = \text{Span}\{e_1\}$, and then the condition of compatibility (12) is equivalent to that $\text{Ker}(C_1)$ is a one-dimensional invariant subspace of $A$:

$$A\text{Ker}(C_1) \subseteq \text{Ker}(C_1).$$ (15)
The condition of compatibility (12), often called the condition of $C_1$-compatibility in what follows, is also equivalent to that there exists a unique matrix $\overline{A}_1$ of order $(N - 1)$, such that

$$C_1 A = \overline{A}_1 C_1. \quad (16)$$

Such matrix $\overline{A}_1$ is called the **reduced matrix of** $A$ **by** $C_1$. 
Under the condition of $C_1$-compatibility, let

$$W_1 = (w^{(1)}, \ldots, w^{(N-1)})^T = C_1U. \quad (17)$$

It is easy to see that the original system (5) for the variable $U$ can be reduced to the following self-closed system for the variable $W_1$:

$$\begin{align*}
W''_1 - \Delta W_1 + \bar{A}_1 W_1 &= 0 \quad \text{in } (0, +\infty) \times \Omega, \\
W_1 &= 0 \quad \text{on } (0, +\infty) \times \Gamma_0, \\
W_1 &= C_1DH \quad \text{on } (0, +\infty) \times \Gamma_1. 
\end{align*}$$

(18)
Under the condition of $C_1$-compatibility, the exact boundary synchronization of the original system for $U$ is equivalent to the exact boundary null controllability of the reduced system for $W$. 
Assume that the condition of $C_1$-compatibility is satisfied and the domain $\Omega$ satisfies the usual multiplier geometrical condition, then there exists a suitably large $T > 0$ such that for any given boundary control matrix $D$ with rank $(C_1D) = N - 1$, the exact boundary synchronization of system (5) can be realized at the time $t = T$.

On the contrary, if rank$(C_1D) < N - 1$, in particular, if rank$(D) < N - 1$, i.e., the number of boundary controls is fewer than $(N - 1)$, then no matter how large $T > 0$ is, the exact boundary synchronization can never be achieved for all initial data $(\hat{U}_0, \hat{U}_1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$. 
The above condition of $C_1$-compatibility is not only sufficient but also necessary to ensure the exact boundary synchronization. Under this condition, for the boundary control matrix $D$ such that \( \text{rank}(D) = \text{rank}(C_1D) = N - 1 \), appropriately chosen \((N - 1)\) boundary controls suffice to meet the requirement.
In the study of synchronization for systems governed by ODEs, the row-sum condition (12) is imposed to the system according to physical meanings as a reasonable sufficient condition. However, for our systems governed by PDEs and for the synchronization on a finite time interval, it is actually a necessary condition, which makes the theory of synchronization more complete for systems governed by PDEs.
In the case that system (5) possesses the exact boundary synchronization at the moment \( T > 0 \), as \( t \geq T \), the exactly synchronizable state \( u = u(t, x) \) satisfies

\[
\begin{cases}
  u'' - \Delta u + au = 0 & \text{in } (T, +\infty) \times \Omega, \\
  u = 0 & \text{on } (T, +\infty) \times \Gamma,
\end{cases}
\]

(19)

where \( a \) is given by the row-sum condition (12). Thus, the evolution of the exactly synchronizable state \( u = u(t, x) \) with respect to \( t \) can be uniquely determined by its initial data:

\[
t = T : \ u = \hat{u}_0, \ u' = \hat{u}_1.
\]

(20)
Generally speaking, the value $\hat{(u_0, u_1)}$ of $(u, u')$ at $t = T$ should depend on the original initial data $(\hat{U}_0, \hat{U}_1)$ as well as on the boundary controls $H$ which realize the exact boundary synchronization. Moreover, the value of $(\hat{u}_0, \hat{u}_1)$ at $t = T$ for a given initial data $(\hat{U}_0, \hat{U}_1)$ can be determined, and in some special cases it is independent of boundary controls $H$ which realize the exact boundary synchronization.
The attainable set of all possible values of \((\hat{u}_0, \hat{u}_1)\) at \(t = T\) is the whole space \(L^2(\Omega) \times H^{-1}(\Omega)\), when the original initial data \((\hat{U}_0, \hat{U}_1)\) vary in the space \((L^2(\Omega))^N \times (H^{-1}(\Omega))^N\). That is to say, any given \((\hat{u}_0, \hat{u}_1)\) in \(L^2(\Omega) \times H^{-1}(\Omega)\) can be the value of an exactly synchronizable state \((u, u')\) at \(t = T\).
4. Exact boundary synchronization by \( p \)-groups
How will be the situation if the number of boundary controls is further reduced? Then we can only further lower the standard to be reached.
One possible way is not to require synchronization of all state variables, but to divide them into several groups, e.g., $p$ groups ($p \geq 1$), and then demand synchronization of state variables within every group, whereby we get the concept of exact boundary synchronization by $p$-groups.
Let $p \geq 1$ be an integer and let

$$0 = n_0 < n_1 < n_2 < \cdots < n_p = N.$$  \hfill (21)

We rearrange the components of $U$ into $p$ groups

$$(u^{(1)}, \cdots, u^{(n_1)}), (u^{(n_1+1)}, \cdots, u^{(n_2)}),$$  
$$\cdots, (u^{(n_{p-1}+1)}, \cdots, u^{(n_p)}).$$  \hfill (22)

Here, to simplify the statement in what follows, we always suppose that the number of state variables in each group is bigger than or equal to 2.
System (5) is **exactly synchronizable by $p$-groups** at the time $T > 0$ if, for any given initial data $(\hat{U}_0, \hat{U}_1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$, there exist suitable boundary controls $H \in (L^2_{loc}(0, +\infty; L^2(\Gamma_1)))^M$ with compact support in $[0, T]$, such that the corresponding solution $U = U(t, x)$ satisfies the following final conditions:

\[
\begin{align*}
  t \geq T: \quad \left\{ 
    u^{(1)} &\equiv \cdots \equiv u^{(n_1)} := u_1, \\
    u^{(n_1+1)} &\equiv \cdots \equiv u^{(n_2)} := u_2, \\
    \cdots & \\
    u^{(n_{p-1}+1)} &\equiv \cdots \equiv u^{(n_p)} := u_p,
  \end{align*}
\]  

(23)

where $u = (u_1, \cdots, u_p)^T$ is called the **exactly synchronizable state by $p$-groups**, which is a **priori** unknown.
Let $S_r$ be the following $(n_r - n_{r-1} - 1) \times (n_r - n_{r-1})$ matrix:

$$S_r = \begin{pmatrix}
1 & -1 & 0 & \cdots & 0 \\
0 & 1 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & -1
\end{pmatrix},$$

(24)
and let $C_p$ be the following $(N - p) \times N$ full row-rank matrix of synchronization by $p$-groups:

$$C_p = \begin{pmatrix}
S_1 & & \\
gap & S_2 & \\
gap & \gap \gap & \ddots \\
gap & & \gap & S_p
\end{pmatrix}.$$ 

(25)
The exact boundary synchronization by $p$-groups (23) is equivalent to

$$t \geq T : \quad C_pU \equiv 0. \quad (26)$$
We can establish all the previous results in a similar way after having overcome some technical difficulties. For instance, if the exact boundary synchronization by $p$-groups can be realized by means of $(N - p)$ boundary controls, we can get the corresponding condition of $C_p$-compatibility: There exists a unique reduced matrix $\overline{A}_p$ of order $(N - p)$, such that

$$C_p A = \overline{A}_p C_p,$$  \hspace{1cm} (27)

which means that the coupling matrix $A$ satisfies the row-sum condition by blocks.
Under the condition of $C_p$-compatibility and the domain $\Omega$ satisfies the usual multiplier geometrical condition, there exists a suitably large $T > 0$, such that for any given boundary control matrix $D$ with rank $(C_pD) = N - p$, the exact boundary synchronization by $p$-groups of system (5) can be realized at the time $t = T$. On the contrary, if rank$(C_pD) < N - p$, in particular, if rank$(D) < N - p$, then no matter how large $T > 0$ is, the exact boundary synchronization by $p$-groups can never be achieved for all initial data $(\hat{U}_0, \hat{U}_1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$. 
### Table 1: The exact boundary synchronization by $p$-groups

<table>
<thead>
<tr>
<th>Condition of $C_p$-compatibility</th>
<th>Minimal number of boundary controls</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact boundary null controllability</td>
<td>$N$</td>
</tr>
<tr>
<td>Exact boundary synchronization</td>
<td>$C_1 A = \overline{A}_1 C_1$</td>
</tr>
<tr>
<td>Exact boundary synchronization by 2-groups</td>
<td>$C_2 A = \overline{A}_2 C_2$</td>
</tr>
<tr>
<td>Exact boundary synchronization by $p$-groups</td>
<td>$C_p A = \overline{A}_p C_p$</td>
</tr>
</tbody>
</table>
What can we do when the number of boundary controls that can be chosen further decreases?
The aforementioned controllability and synchronization should both be established in the exact sense, however, from the practical point of view, some minor error will not affect the general situation, so it also makes sense that these requirements are tenable only in the approximate sense.
What we are considering here is the situation that no matter how small the error given beforehand is, we can always find suitable boundary controls so that the controllability or synchronization can be realized within the permitted range of accuracy. Since the error can be chosen progressively smaller, this corresponds actually to a limit process, which thus allows the analytical method to be applied more effectively. The corresponding controllability and synchronization are called **approximate boundary null controllability** and **approximate boundary synchronization**.
5. Approximate boundary null controllability
System (5) possesses the **approximate boundary null controllability** at the moment $T > 0$, if for any given initial data $(\hat{U}_0, \hat{U}_1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$, there exists a sequence $\{H_n\}$ of boundary controls, $H_n \in (L^2_{loc}(0, +\infty; L^2(\Gamma_1)))^M$ with compact support in $[0, T]$, such that the corresponding sequence $\{U_n\}$ of solutions to problem (5)-(6) satisfies the following condition:

$$U_n \longrightarrow 0$$

in $C^0_{loc}([T, +\infty); (L^2(\Omega))^N) \cap C^1_{loc}([T, +\infty); (H^{-1}(\Omega))^N)$ as $n \rightarrow +\infty$. 

Li Tatsien  ( Fudan University)
November 12, 2018  72 / 146
The exact boundary null controllability leads to the approximate boundary null controllability. However, the approximate boundary null controllability can not lead to the exact boundary null controllability in general.
Consider the corresponding adjoint problem

\[
\begin{aligned}
\Phi'' - \Delta \Phi + A^T \Phi &= 0 & \text{in } (0, +\infty) \times \Omega, \\
\Phi &= 0 & \text{on } (0, +\infty) \times \Gamma, \\
t = 0 : \Phi = \hat{\Phi}_0, \Phi' = \hat{\Phi}_1 & \text{in } \Omega,
\end{aligned}
\]

(29)

where $A^T$ denotes the transpose of $A$. 
The adjoint problem (29) is $D$-observable on the interval $[0, T]$, if

$$D^T \partial_\nu \Phi \equiv 0 \quad \text{on} \quad [0, T] \times \Gamma_1 \quad \Longrightarrow \quad (\widehat{\Phi}_0, \widehat{\Phi}_1) \equiv 0,$$

i.e., $\Phi \equiv 0$, (30)

where $\partial_\nu$ denotes the outward normal derivative on the boundary.
The $D$-observability is only a weak observability in the sense of uniqueness, which can not guarantee the exact boundary null controllability.

However, System (5) is approximately null controllable at the moment $T > 0$ if and only if the adjoint system (29) is $D$-observable on the interval $[0, T]$. 
If $M = N$, then by Holmgren’s uniqueness theorem, system (5) is always approximately null controllable by means of $N$ boundary controls for $T > 0$ large enough, even without the multiplier geometrical condition.

The approximate boundary null controllability can be realized even if $M < N$, namely, by means of fewer boundary controls.
To transform the approximate boundary null controllability of the original system equivalently to the $D$-observability of the adjoint problem cannot solve specifically the problem of judging whether a system possesses the approximate boundary null controllability, nor is it clear to what extent the weakened concept may reduce the number of boundary controls needed.
Assuming that system (5) is approximately null controllable at the time $T > 0$, then we necessarily have that the following enlarged matrix composed of $A$ and $D$ is full rank:

$$\text{rank}(D, AD, \cdots, A^{N-1}D) = N. \quad (31)$$

This is a necessary condition which helps to conveniently eliminate a set of systems that do not meet the requirements.
(31) is nothing but the so-called Kalman’s criterion for guaranteeing the exact controllability for the following system of ODEs:

\[ X' = AX + Du, \quad (32) \]

where \( u \) stands for the vector of control variables, however here we get it from a different point of view.
Since condition (31) is independent of $T$, it is not a sufficient condition of the $D$-observability for adjoint problem (29) in general, otherwise the $D$-observability should be realized almost immediately, but it is not the case because of the finite speed of wave propagation.
Under certain assumptions on $A$, condition (31) is also sufficient for guaranteeing the approximate boundary null controllability for $T > 0$ large enough for some special kinds of systems, for instance, some one-space-dimensional systems, some $2 \times 2$ systems, the cascade system and more generally, the nilpotent system under the multiplier geometrical condition, etc.
For the exact boundary null controllability of system (5), the number $M = \text{rank}(D)$, namely, the number of boundary controls, should be equal to $N$, the number of state variables. However, the approximate boundary null controllability of system (5) could be realized if the number $M = \text{rank}(D)$ is substantially small, even if $M = \text{rank}(D) = 1$. Nevertheless, even if the rank of $D$ might be small, but because of the existence and influence of the coupling matrix $A$, in order to realize the approximate boundary null controllability, the rank of the enlarged matrix $(D, AD, \ldots, A^{N-1}D)$ should be still equal to $N$, the number of state variables.
The rank $M$ of $D$ is the number of "direct" boundary controls acting on $\Gamma_1$, and $\text{rank}(D, AD, \cdots, A^{N-1}D)$ denotes the number of "total" controls. Differently from the exact boundary null controllability, for the approximate boundary null controllability, we should consider not only the number of direct boundary controls, but also the number of total controls.
Example. 1 We consider the following coupled system of wave equations:

\[
\begin{align*}
\frac{\partial^2 u_1}{\partial t^2} - \Delta u_1 + a_2 u_2 &= 0, \\
\frac{\partial^2 u_2}{\partial t^2} - \Delta u_2 + a_3 u_3 &= 0, \\
&\vdots \\
\frac{\partial^2 u_{N-1}}{\partial t^2} - \Delta u_{N-1} + a_N u_N &= 0, \\
\frac{\partial^2 u_N}{\partial t^2} - \Delta u_N &= 0,
\end{align*}
\]

in which \(a_i \neq 0\) (\(i = 2, \cdots, N\)), 
and the coupling matrix

\[
A = \begin{pmatrix}
0 & a_2 & 0 & \cdots & 0 & a_N \\
0 & 0 & a_3 & \cdots & 0 & a_{N-1} \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & a_N \end{pmatrix}
\]

is of cascade type.
In particular, we take $\mathcal{M} = 1$ with only one boundary control function $h(t)$, and the boundary control matrix

$$D = (0, 0, \cdots, 0, 1)^T.$$ 

Then the boundary condition on $\Gamma_1$ is written as

$$u_1 = \cdots = u_{N-1} = 0 \quad \text{and} \quad u_N = h(t) \quad \text{on} \quad \Gamma_1$$

and the observation for the adjoint problem (29) is

$$D^T \frac{\partial \Phi}{\partial \nu} = \frac{\partial \varphi_N}{\partial \nu} \equiv 0 \quad \text{on} \quad \Gamma_1.$$
By a result of F. Alabau, under the multiplier geometrical condition, the corresponding adjoint system is D-observable on the interval \([0, T]\) with \(T > 0\) large enough. Then, the original system is approximately null controllable at the moment \(T > 0\). Thus, by means of only one boundary control \(h\) we get the approximate null controllability.
Moreover, we can prove that for any given $N \times M$ boundary control matrix $D$ such that the corresponding Kalman’s criterion (31) is satisfied, similar results hold. Thus, the Kalman’s criterion (31) is sufficient for the coupled system of cascade type.
Even only one boundary control can be used to realize the approximate null controllability for the couple system of cascade type, however, the choice of such a boundary control can not be arbitrary. To illustrate this, we take

\[ D = (1, 0, \cdots, 0)^T, \quad (0, 1, 0, \cdots, 0)^T, \quad \cdots, \quad \text{or} \quad (0, \cdots, 0, 1, 0)^T, \]

respectively, and the corresponding observation for the adjoint problem is then

\[ D^T \frac{\partial \Phi}{\partial \nu} = \frac{\partial \varphi_1}{\partial \nu}, \quad \frac{\partial \varphi_2}{\partial \nu}, \quad \cdots, \quad \text{or} \quad \frac{\partial \varphi_{N-1}}{\partial \nu} \equiv 0 \quad \text{on} \quad \Gamma_1, \]

which can not guarantee the \( D \)-observability.
This situation still holds even more boundary controls are used. For instance, we take $M = N - 1$, $H(t) = (h_1(t), \ldots, h_{N-1}(t))^T$ and

$$D = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \ldots & 1 \\
0 & 0 & \ldots & 0 \\
\end{pmatrix}_{N \times (N-1)}.$$

Then the observation for the adjoint problem is

$$D^T \frac{\partial \Phi}{\partial \nu} = \left( \frac{\partial \varphi_1}{\partial \nu}, \ldots, \frac{\partial \varphi_{N-1}}{\partial \nu} \right)^T \equiv 0 \quad \text{on } \Gamma_1.$$

In this case, $(N - 1)$ boundary controls $h_1(t), \ldots, h_{N-1}(t)$ are still useless for the approximate boundary null controllability.

The reason is very simple: the Kalman’s criterion fails in these cases.
6. Approximate boundary synchronization
System (5) possesses the approximate boundary synchronization at the moment $T > 0$, if for any given initial data $(\hat{U}_0, \hat{U}_1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$, there exist a sequence $\{H_n\}$ of boundary controls, $H_n \in (L^2_{loc}(0, +\infty; L^2(\Gamma_1)))^M$ with compact support in $[0, T]$, such that the corresponding sequence $\{U_n\} = \{(u_n^{(1)}, \cdots, u_n^{(N)})^T\}$ of solutions to problem (5)-(6) satisfies

$$u_n^{(k)} - u_n^{(l)} \to 0$$

in $C^0_{loc}([T, +\infty); L^2(\Omega)) \cap C^1_{loc}([T, +\infty); \times H^{-1}(\Omega))$ as $n \to +\infty$ for all $1 \leq k, l \leq N$. 
If system (5) is exactly synchronizable, then it must be approximately synchronizable; however, the inverse is not true in general.

Moreover, the approximate boundary null controllability obviously leads to the approximate boundary synchronization. We should exclude this trivial situation in advance.
Assume that system (5) is approximately synchronizable, but not approximately null controllable. Then, as in the case of exact boundary synchronization, the coupling matrix $A$ should satisfy the same condition of $C_1$-compatibility (12).
Setting $W_1 = C_1 U$ as in (17), we get again the reduced system (18) and its adjoint problem (the **reduced adjoint problem**):

\[
\begin{cases}
\Psi''_1 - \Delta \Psi_1 + A_1^T \Psi_1 = 0 & \text{in } (0, +\infty) \times \Omega, \\
\Psi_1 = 0 & \text{on } (0, +\infty) \times \Gamma, \\
 t = 0 : (\Psi_1, \Psi'_1) = (\hat{\Psi}_0, \hat{\Psi}_1) & \text{in } \Omega.
\end{cases}
\]

Similarly to the $D$-observability, we say that the reduced adjoint problem (34) is $C_1 D$-**observable** on the interval $[0, T]$, if

\[
(C_1 D)^T \partial_{\nu} \Psi \equiv 0 \text{ on } [0, T] \times \Gamma_1
\]

\[\implies (\hat{\Psi}_0, \hat{\Psi}_1) \equiv 0, \text{ i.e., } \Psi \equiv 0.\]
Under the condition of $C_1$-compatibility, system (5) is approximately synchronizable at the moment $T > 0$, if and only if the reduced adjoint problem (34) is $C_1D$-observable on the interval $[0, T]$. 
Under the condition of $C_1$-compatibility, if \( \text{rank}(C_1D) = N - 1 \) (which implies \( M \geq N - 1 \)), then, system (5) is always approximately synchronizable, even without the multiplier geometrical condition.
Even if \( \text{rank}(C_1D) < N - 1 \), and in particular, if we essentially use fewer than \((N - 1)\) boundary controls, it is still possible to realize the approximate boundary synchronization.
As in the case of approximate boundary null controllability, under the condition of $C^1$-compatibility, we have: Assume that system (5) is approximately synchronizable at the time $T > 0$, then we necessarily have

$$\text{rank}(C_1 D, C_1 AD, \cdots, C_1 A^{N-1} D) = N - 1.$$  \hfill (36)

Condition (36) is not sufficient in general for the approximate boundary synchronization, however, it is still sufficient for $T > 0$ large enough for some special systems under certain additional assumptions on $A$. 
Assume that system (5) is approximately synchronizable under the action of a boundary control matrix $D$. No matter whether the condition of $C_1$-compatibility is satisfied or not, we necessarily have

$$\text{rank}(D, AD, \cdots, A^{N-1}D) \geq N - 1,$$

namely, at least $(N - 1)$ total controls are needed in order to realize the approximate boundary synchronization of system (5).
Assume that system (5) is approximately synchronizable under the minimal rank condition

\[ \text{rank}(D, AD, \cdots, A^{N-1}D) = N - 1. \]  

(38)

Then the coupling matrix \( A \) should satisfy the condition of \( C_1 \)-compatibility and some algebraic properties, and there exists a scalar function \( u \) as the approximately synchronizable state, such that

\[ u_n^{(k)} \to u \text{ in } C^0_{loc}([T, +\infty); L^2(\Omega)) \cap C^1_{loc}([T, +\infty); H^{-1}(\Omega)) \]  

(39)

as \( n \to +\infty \) for all \( 1 \leq k \leq N \). Moreover, the approximately synchronizable state \( u \) is independent of the sequence \( \{H_n\} \) of the employed controls. Thus the original approximate boundary synchronization in the consensus sense reduces to that in the pinning sense.
Example. 2 For some special systems of \( N \) wave equations, even if \( N \) is quite large, only one suitable boundary control can be used to realize the approximate boundary synchronization.

For instance, consider the following coupled system of four wave equations:

\[
\begin{align*}
\frac{\partial^2 u_1}{\partial t^2} - \Delta u_1 &= 0, \\
\frac{\partial^2 u_2}{\partial t^2} - \Delta u_2 &= 0, & \text{in } \Omega \\
\frac{\partial^2 u_3}{\partial t^2} - \Delta u_3 - u_1 + u_2 &= 0, \\
\frac{\partial^2 u_4}{\partial t^2} - \Delta u_4 - u_1 + u_3 &= 0, \\
\end{align*}
\]

\[
\begin{align*}
u_1 &= u_2 = u_3 = u_4 = 0 & \text{on } \Gamma_0, \\
u_1 &= h^{(1)}, u_2 = u_3 = u_4 = 0 & \text{on } \Gamma_1.
\end{align*}
\]
Here, the coupling matrix

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
\end{pmatrix}
\]

satisfies the row-sum condition, namely, the condition of \(C_1\)-compatibility, and the reduced matrix \(\overline{A}\) is of the cascade type:

\[
\overline{A} = \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{pmatrix}.
\]
Moreover,

\[
D = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad C_1 D = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.
\]

Since the reduced system is just of the cascade form, the corresponding Kalman’s criterion is not only necessary, but also sufficient for the approximate boundary synchronization at the time \( T > 0 \), provided that the multiplier geometrical condition is satisfied and \( T > 0 \) is large enough. Then, by means of only one boundary control \( h^{(1)} \) we get the approximate boundary synchronization.
When we take the boundary condition on $\Gamma_1$ as

$$u_2 = h^{(2)}, \quad u_1 = u_3 = u_4 = 0 \quad \text{on} \quad (0, +\infty) \times \Gamma_1,$$

we have a similar conclusion. However, when we take the boundary condition on $\Gamma_1$ as

$$u_3 = h^{(3)}, \quad u_1 = u_2 = u_4 = 0 \quad \text{on} \quad (0, +\infty) \times \Gamma_1$$

or

$$u_4 = h^{(4)}, \quad u_1 = u_2 = u_3 = 0 \quad \text{on} \quad (0, +\infty) \times \Gamma_1$$

or even

$$u_1 = 0, \quad u_2 = 0, \quad u_3 = h^{(3)}, \quad u_4 = h^{(4)} \quad \text{on} \quad (0, +\infty) \times \Gamma_1$$

e tc., the Kalman’s criterion fails and the approximate boundary synchronization can not be realized.
7. Approximate boundary synchronization by \( p \)-groups
Generally speaking, we can define the approximate boundary synchronization by $p$-groups ($p \geq 1$). Let us regroup the components of the state variable as in (22).
We say that system (5) is approximately synchronizable by $p$-groups at the time $T > 0$, if for any given initial data $(\hat{U}_0, \hat{U}_1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$, there exists a sequence $\{H_n\}$ of boundary controls in $(L^2_{loc}(0, +\infty; L^2(\Gamma_1)))^M$ with compact support in $[0, T]$, such that the sequence $\{U_n\}$ of the corresponding solutions satisfies the following conditions:

$$u_n^{(k)} - u_n^{(l)} \to 0$$

in $C^0_{loc}([T, +\infty); L^2(\Omega)) \cap C^1_{loc}([T, +\infty); H^{-1}(\Omega))$ for all $n_{r-1} + 1 \leq k, l \leq n_r; \ 1 \leq r \leq p$, or, equivalently

$$C_p U_n \to 0$$

in $C^0_{loc}([T, +\infty); (L^2(\Omega))^{N-p}) \cap C^1_{loc}([T, +\infty); (H^{-1}(\Omega))^{N-p})$ as $n \to +\infty$, in which $C_p$ is given by (25).
As in the case of exact boundary synchronization by $p$-groups, we can give the corresponding condition of $C_p$-compatibility: There exists a unique reduced matrix $\overline{A}_p$ of order $(N - p)$, such that (27) holds. Under the condition of $C_p$-compatibility, we necessarily get the following Kalman’s criterion:

$$\text{rank}(C_pD, C_pAD, \cdots, C_pA^{N-1}D) = N - p.$$  \hspace{1cm} (42)
Assume that system (5) is approximately synchronizable by $p$-groups under the action of a boundary control matrix $D$. Then, no matter whether the condition of $C_p$-compatibility is satisfied or not, we necessarily have

$$\text{rank}(D, AD, \ldots, A^{N-1} D) \geq N - p,$$

namely, at least $(N - p)$ total controls are needed in order to realize the approximate boundary synchronization of system (5).
Assume that system (5) is approximately synchronizable by $p$-groups under the minimal rank condition
\[ \text{rank}(D, AD, \ldots, A^{N-1}D) = N - p. \]  
(44)

Then the coupling matrix $A$ should satisfy the condition of $C_p$-compatibility and $\text{Ker}(C_p)$ admits a supplement $V$, which is also invariant for $A$. Moreover, there exist some linearly independent scalar functions $u_1, \cdots, u_p$ such that the approximately synchronizable state $u = (u_1, \cdots, u_p)^T$ is independent of the sequence $\{H_n\}$ of employed controls. Thus the original approximate synchronization by $p$-groups in the consensus sense reduces to that in the pinning sense.
### Table 2: The approximate boundary synchronization by $p$-groups

<table>
<thead>
<tr>
<th>Approximate boundary</th>
<th>Condition of $C_p$-compatibility</th>
<th>Minimal number of total controls</th>
</tr>
</thead>
<tbody>
<tr>
<td>null controllability</td>
<td></td>
<td>$N$</td>
</tr>
<tr>
<td>synchronization</td>
<td>$C_1 A = \overline{A}_1 C_1$</td>
<td>$N - 1$</td>
</tr>
<tr>
<td>synchronization by 2-groups</td>
<td>$C_2 A = \overline{A}_2 C_2$</td>
<td>$N - 2$</td>
</tr>
<tr>
<td></td>
<td>$\ldots$</td>
<td></td>
</tr>
<tr>
<td>$p$-groups</td>
<td>$C_p A = \overline{A}_p C_p$</td>
<td>$N - p$</td>
</tr>
</tbody>
</table>
Example. 3  We will give a system which is approximately boundary synchronizable by 2-groups, but is not approximately boundary synchronizable.
Consider the system

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - \Delta u + \varepsilon w &= 0, \\
\frac{\partial^2 v}{\partial t^2} - \Delta v + \varepsilon z &= 0, \\
\frac{\partial^2 w}{\partial t^2} - \Delta w + \varepsilon u &= 0, \\
\frac{\partial^2 z}{\partial t^2} - \Delta z + \varepsilon v &= 0,
\end{align*}
\]

\[(u, v, w, z)^T|_{\Gamma_0} = 0, \quad (u, v, w, z)^T|_{\Gamma_1} = Dh(t),\]

where $\varepsilon > 0$ is a parameter with $|\varepsilon|$ small enough.

$D = (1, 0, 0, 0)^T$ and $h(t)$ is a boundary control.
\( N = 4, \ M = 1, \) and the coupling matrix

\[
A = \begin{pmatrix}
0 & 0 : \varepsilon & 0 \\
0 & 0 : 0 & \varepsilon \\
\varepsilon & 0 : 0 & 0 \\
0 & \varepsilon : 0 & 0 \\
\end{pmatrix}
\]

satisfies not only the condition of \( C_1 \)-compatibility for the synchronization (the sum of all elements of every row is equal to \( \varepsilon \)), but also the condition of \( C_2 \)-compatibility for the synchronization by 2-groups (as a \( 2 \times 2 \) block matrix, the sum of all elements of every row for each block is equal to \( \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix} \)).
The corresponding matrices for the synchronization and for the synchronization by 2-groups are

\[ C_1 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \]

respectively.

Moreover, the reduced matrix of A by \( C_2 \) is a \( 2 \times 2 \) symmetric matrix

\[ \bar{A}_2 = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}. \]
Since the corresponding Kalman’s criterion fails, it is impossible to get the approximate boundary synchronization by means of this boundary control $h(t)$. 
However, the Kalman’s criterion for approximate boundary synchronization by 2-groups is satisfied, and under the multiplier geometrical condition, for $\varepsilon > 0$ small enough, this condition is not only necessary, but also sufficient. So we can get the approximate boundary synchronization by 2-groups by means of one boundary control $h(t)$. 
8. Induced approximate synchronization
We can not always realize the approximate boundary synchronization by $p$-groups under the minimal rank condition (44). In fact, let $\mathbb{D}_p$ be the set of all matrices $D$ which realize the approximate boundary synchronization by $p$-groups for system (5). Define the minimal number $N_p$ of total controls by

$$N_p = \inf_{D \in \mathbb{D}_p} \text{rank}(D, AD, \cdots, A^{N-1}D).$$

(45)

We can prove

$$N_p = N - q,$$

(46)

where $q \leq p$ is the dimension of the largest subspace $W$, which is contained in $\text{Ker}(C_p)$ and admits a supplement $V$, both $W$ and $V$ being invariant for the matrix $A$. 
So, generally speaking, \((N - q)\) total controls are necessary for the approximate boundary synchronization by \(p\)-groups of system (5), while (42) provides only the convergence of \((N - p)\) components of state variables. Hence, there is a loss of \((p - q)\) information hidden in the minimal rank condition

\[
\text{rank}(D, AD, \cdots, A^{N-1}D) = N - q. \quad (47)
\]
Let $C_q^*$ be the induced extension matrix defined by $\text{Ker}(C_q^*) = W$. Then, $\text{Ker}(C_q^*)$ is the biggest subspace of $A$, which is contained in $\text{Ker}(C_p^*)$ and admits a supplement $V$, both are invariant for $A$. Moreover, we have the following rank condition:

$$\text{rank}(C_q^* D, C_q^* AD, \cdots, C_q^* A^{N-1} D) = N - q.$$  

(48)
In some specific situations, we have

\[ C^*_q U_n \to 0 \quad (49) \]

In this case, we say that system (5) is induced approximately synchronizable.

\[ C^0_{loc}([T, +\infty); (L^2(\Omega))^{N-q}) \cap C^1_{loc}([T, +\infty); (H^{-1}(\Omega))^{N-q}) \text{ as } n \to +\infty. \]
From the induced approximate synchronization, we can get that there exist some scalar functions $u_1, \cdots, u_p$ independent of the sequence $\{H_n\}$ of employed controls, such that

$$u^{(k)}_n \rightarrow u_r$$

in $C^0_{loc}([T, +\infty); L^2(\Omega)) \cap C^1_{loc}([T, +\infty); H^{-1}(\Omega))$ as $n \rightarrow +\infty$ for all $n_r - 1 + 1 \leq k \leq n_r; 1 \leq r \leq p$. Thus, the approximate boundary synchronization by $p$-groups in the consensus sense is in fact in the pinning sense. Nevertheless, unlike the case $N_p = N - p$, these functions $u_1, \cdots, u_p$ are linearly dependent.
Example. 4 Let $N = 4, M = 1, p = 2$, and

$$A = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$ 

We have the condition of $C_2$-compatibility.
The corresponding Kalman’s criterion is satisfied, and it is not only necessary but also sufficient for the approximate boundary synchronization by 2-groups, since the reduced matrix of $A$ by $C_2$ is of cascade type:

$$\bar{A}_2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$
Then, the original system is approximate synchronizable by 2-groups under the action of the boundary control matrix $D$:

$$C_2U_n \longrightarrow 0, \quad \text{as} \quad n \rightarrow +\infty.$$
In this case, the number of total controls, namely, the rank of the matrix \((D, AD, A^2D, A^3D)\), is equal to 3, which is bigger than \(N - p = 2\). We can actually get the induced approximate synchronization, which implies the following much precise conclusion:

There exists a non-trivial function \(u\) such that

\[
\begin{align*}
    u_n^{(1)} & \rightarrow u, & u_n^{(2)} & \rightarrow u \\
    u_n^{(3)} & \rightarrow 0, & u_n^{(4)} & \rightarrow 0
\end{align*}
\]

as \(n \rightarrow +\infty\).
9. Prospects
The study of exact and approximate boundary synchronizations has just begun, and there are still many problems worthy of further research.
1. We have discussed above the coupled system of wave equations with Dirichlet boundary controls. But we can still consider further the system with Neumann boundary controls

\[
\frac{\partial U}{\partial \nu} = DH(t) \quad \text{on } \Gamma_1 
\]  

(51)

or with the following coupled Robin boundary controls.

\[
\frac{\partial U}{\partial \nu} + BU = DH(t) \quad \text{on } \Gamma_1, 
\]  

(52)

in which \(\partial_\nu\) denotes the outward normal derivative, and \(B = (b_{ij})\) is the boundary coupling matrix of order \(N\) with constant elements.
Since the solution to the corresponding problem with Neumann or coupled Robin boundary conditions has less regularity than that with Dirichlet boundary conditions, moreover, there is a second coupling matrix $B$ in the coupled Robin boundary conditions, technically speaking, we will encounter more difficulties and some different and new treatments are necessarily needed.
2. The above research is limited to the linear situation, and the method employed is also linear. However, the study of synchronization should be extended to the nonlinear situation. At present, in the one-space-dimensional case, for the coupled system of quasilinear wave equations with various boundary controls, the exact boundary synchronization has been achieved in the framework of classical solutions, and there is still larger developing space for more general research.
3. In addition to the study of exact boundary controllability of hyperbolic systems on the whole space domain, in the recent years, due to the demands of applications, in the one-space-dimensional situation, the research on exact boundary controllability of hyperbolic systems at one node has been developed, which is called the **exact boundary controllability of nodal profile**. Correspondingly, the investigation of approximate boundary controllability of nodal profile as well as exact and approximate boundary synchronizations of nodal profile should also be carried further.
4. The study above focuses on the exact or approximate synchronization for a coupled system of wave equations in finite time through boundary controls. It is also worthwhile to study the asymptotic synchronization in the linear and nonlinear situations without any control when $t \to +\infty$. This should be a meaningful extension of the research on the asymptotic stability of the solution to the coupled system of wave equations.
5. The coupled system of first-order linear or quasilinear hyperbolic systems has wider implications than the coupled system of wave equations, and to do the study of similar problems on it will be of great significance, though more difficult.
6. We can also probe into similar problems on a complicate domain formed by networks. The corresponding results on the exact boundary controllability and on the exact boundary controllability of nodal profile have been done, the synchronization on a network is certainly an interesting problem.
7. Some essentially new results can be obtained if we study the phenomena of synchronization through coupling among individuals with possibly different motion laws (governing equations). For the moment, we just call such type of synchronization \textit{singular synchronization} whose nature is yet to be explored. Recently, the research on the existence of the exactly synchronizable state for a coupled system of wave equations with different wave speeds has been initiated.
8. The stability of the exactly synchronizable state or the approximate synchronizable state should be studied systematically.
9. To extend the concept of synchronization to the case of components with different time delay will be more challenging and may expose quite different features.
10. Under the study on the generalized exact boundary synchronization, it is worth to consider the generalized approximate boundary synchronization.
11. To do similar research on the coupled system of other linear or nonlinear evolution equations (such as beam equations, plate equations, heat equations, etc.) will reveal many new properties and characteristics, which is also quite meaningful.
12. To dig into the practical applications of related results will further enrich the new field of study and lead to greater promotion and influence of the subject.
Finally, on this grand occasion, please allow me to extend my best wishes to Philippe, may good health, happiness and success be with you always! I also hope that the friendship between us, and between mathematicians in China (including Hong Kong) and France can last forever.
Thank you very much