

INTERFACES / JUNCTIONS / STRATIFICATION

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IAS Distinguished Lecture

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I INTRODUCTION

II FNL ELLIPTIC CASE

III HJ (+) CASE

joint project with P. E. SOUGANIDIS

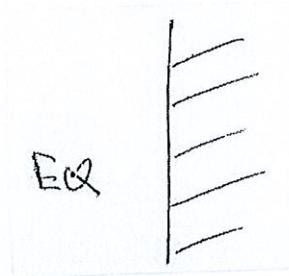
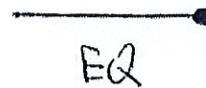
(part of my CdF courses 2015-2016 and 2016-2017)

detailed announcement in Rend. Acad. Lincei 2016

I. INTRODUCTION

I.1 EXS

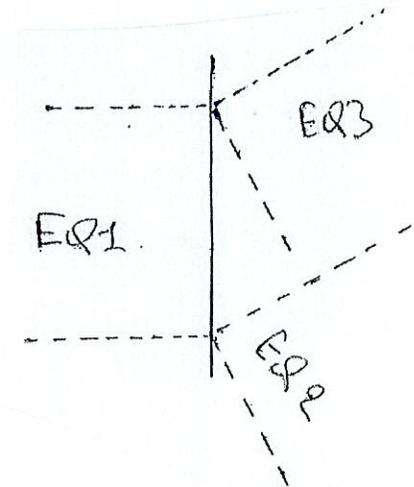
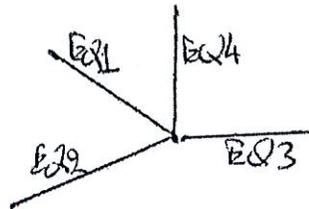
- SOME "SINGULARITIES"



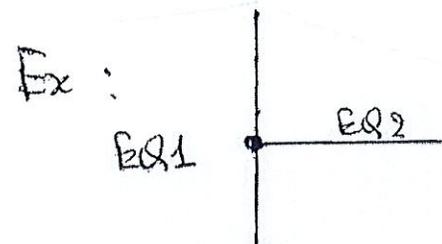
- "INTERFACE"



- "JUNCTION"



- "STRATIFICATION"



Here EQ = VISCOSITY SOL. EQS (SCALAR, MAX. P. , NON CONSERV.).

I.2 MOTIVATIONS

- THIS TYPE OF PROBLEMS ARISE IN **MANY** CONTEXTS: NETWORKS, TRAFFIC, MECHANICS, FLUID MECHANICS, CONTROL, WEBS, DATA, ECONOMICS...
- “HERE”, SAME TYPE OF MODELS ON ALL “SIDES” (# COUPLING OF MODELS), NONLINEAR PROBLEMS WITHOUT VARIATIONAL STRUCTURE (# STRUCTURAL MECHANICS, POSSIBLE CONNECTIONS WITH CIARLET’ SCHOOL)
- TODAY: MAXIMUM PRINCIPLE, SCALAR EQUATIONS, VISCOSITY SOLUTIONS
but scalar conservation laws, wave equations...

I.3 HUGE LITTERATURE (VISCOSITY SOL. TH.)

Soner 86; Dupuis 92;

Garaviello-Soravia 04. 06; Soravia 06; Deler-Soravia 10;

Deckanick-Elliott 04; Coolite-Risebro 07;

Bressan-Yong 07; Blanc 97, 01;

Giga-Gorke-Rybka 11;

Barles-Briani-Chasseigne 13, 14; Barles-Chasseigne 15;

Camilli-Siconolfi 05; Camilli-Marchi 14;

Achdou-Camilli-Cutri-Tchou 13; Camilli-Schreber 13;

Rao-Zidani 13; Rao-Siconolfi-Zlidani 15;

Imbert-Monneau 15; Imbert-Nguyen 16;

...

Achdou, Giraud, Scheinkman, Lasry, Lions (Mining)

Math. Econ /

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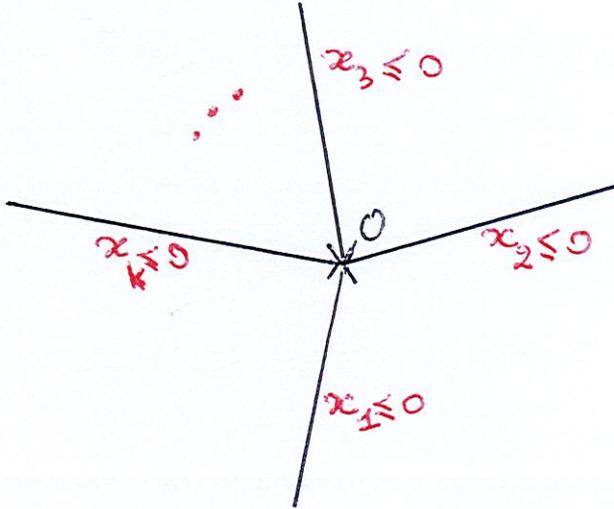
Beyond Rat. Anticip.

...

I.4 OUR GOAL

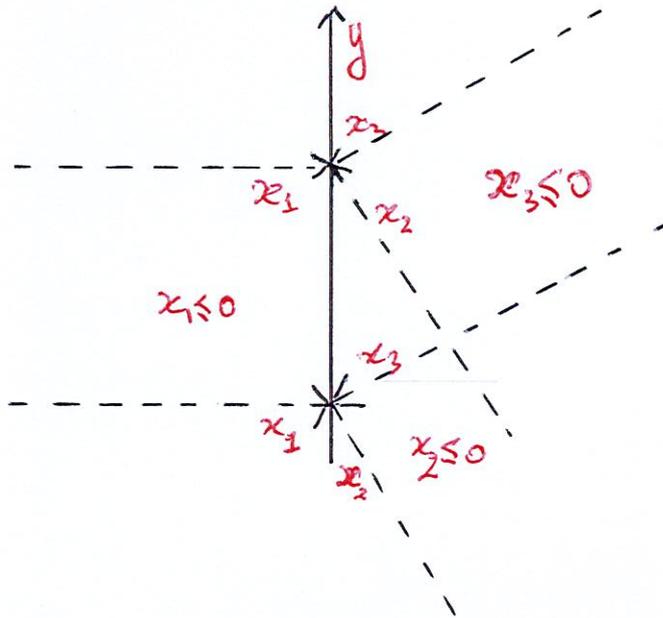
- DESCRIBE A GENERAL (VISCOSITY SOL'S STYLE) PDE THEORY TO STUDY SYSTEMATICALLY ALL THESE PROBLEMS (for general eqs 1st or 2nd order, general nonlinearities non necessarily convex. . .)
- WILL SAMPLE OUR RESULTS AND DESCRIBE A FEW OF THE INGREDIENTS (some of which could be of independent interest. . .)
- HAS LED US TO DISCOVER NEW FACTS AND THEORIES (2nd order nonlinear boundary conditions, boundary behaviour, new approach for conservation laws. . .)

$d = \underline{1}$



$K \geq 2$

$d \geq 1$



II. FNL ELLIPTIC CASE (ID FOR PARABOLIC...)

II.1 NONLINEAR KIRCHOFF CONDITIONS

1D JUNCTIONS

$$u = u_1(x_1), = u_2(x_2), = u_3(x_3) \quad \text{CONT. AT } 0$$

$$\text{For } x_i < 0, F_i \left(x_i, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i^2} \right) = 0$$
$$\begin{array}{ccc} & \uparrow & \uparrow \\ & p & a \end{array}$$

$$\text{ex. } \frac{\partial F_i}{\partial a} \leq -\nu < 0$$

strictly elliptic (F_i Lip in all variables...)

Let $G(p_1, \dots, p_K, z) \nearrow$ in all variables. (Lip)

$$\frac{\partial G}{\partial p_i} \geq \nu > 0$$

Prop. $\exists!$ $u \in C^1(x_i \leq 0)$ s.t. at 0 ($u \in C^{1,1}$)

$$G\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_K}, u\right) = 0$$

“Proof” u subsolution, v supersolution (C^1), $\max u - v$ at 0 > 0

$$\frac{\partial u}{\partial x_i}(0) \geq \frac{\partial v}{\partial x_i}(0), \quad BC \implies \text{equality classical max } p!$$

RKI: FOR LINEAR ELLIPTIC OPERATORS AND A LINEAR BDRY COND.

$$(*) \quad \lambda_0 u + \sum_{i=1}^K \lambda_i \frac{\partial u}{\partial x_i} = 0 \quad \lambda_i > 0, \lambda_0 \geq 0$$

(div. form op., no 1st order terms \longleftrightarrow λ_0 , the whole system is conservative IFF

$$(\lambda_1, \dots, \lambda_K) = \text{Cst} (a_1(0), \dots, a_K(0))$$

where $a_i(x)$ is the diffusion coefficient)

$(\frac{1}{2} \frac{\partial^2}{\partial x^2}$, Brownian motions $(*)$ is related to “splitting probabilities”

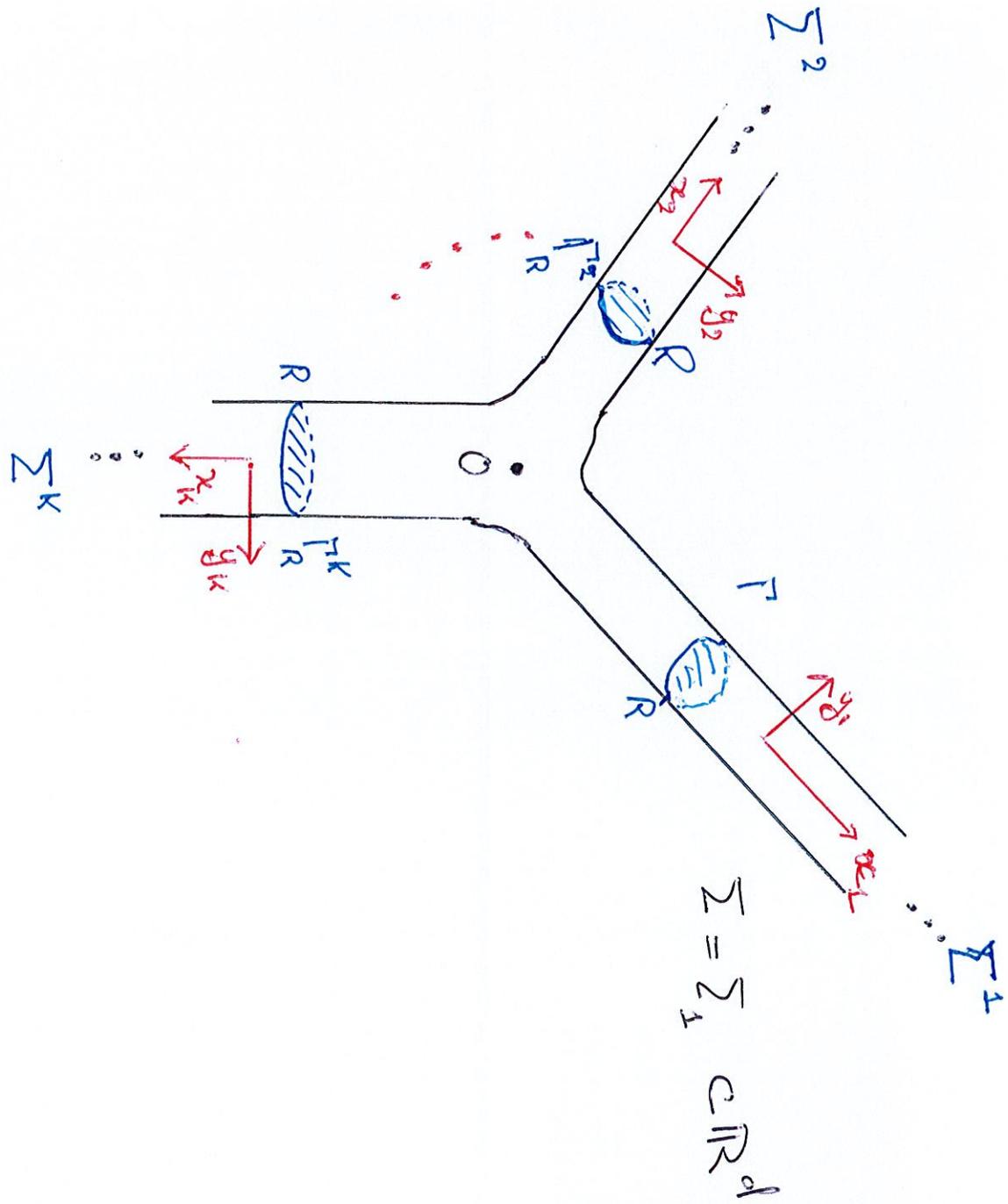
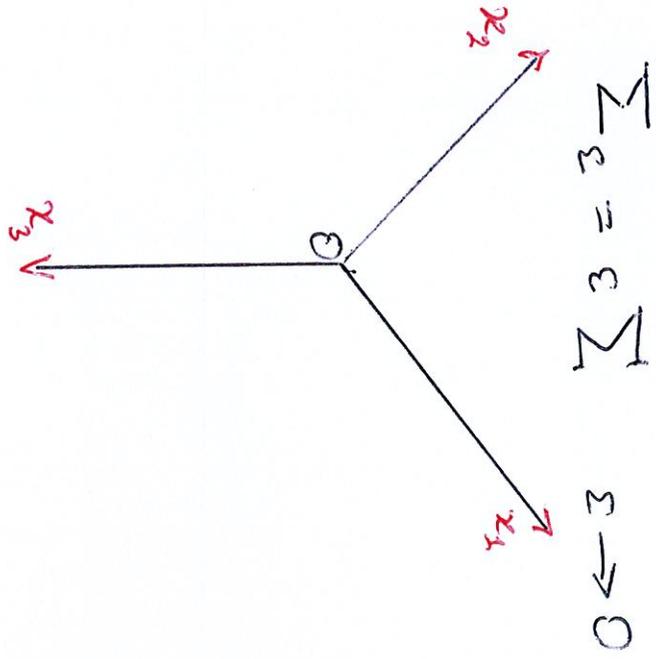
+ λ_0 is related to “waiting” time at 0)

RK2: MULTI D

SAME WITH

$$G\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_{y_2}}, D_y^2 u, D_y u, u, y\right) = 0$$

\nearrow , \dots , \nearrow , \searrow , x , \nearrow , x



II.2 MULTI-SCALE ANALYSIS AND NONLINEAR K. BC

EX:

$$u_\varepsilon + F\left(x, \frac{x}{\varepsilon}, D^2 u_c\right) = 0 \text{ in } \Sigma_\varepsilon$$

$$+ N.BC \quad \frac{\partial u_\varepsilon}{\partial n} = 0 \text{ on } \partial \Sigma_\varepsilon$$

n unit outward normal.

$$F \text{ unifly elliptic: } -\frac{1}{C_0} I \leq \frac{\partial F}{\partial A} \leq -C_0 I \quad (C_0 \geq 1)$$

$F(x, x', A)$ indt of x' on each Σ^i for $|e|$ large

$$(= F_i(x, A))$$

PROBLEM

$$u_\varepsilon \xrightarrow{\varepsilon} ?$$

Simplifying assumption: $\varepsilon F\left(\frac{A}{\varepsilon}\right) \rightarrow \bar{F}(A) \quad \forall x', \text{ at } x = 0$

Ansatz: “ $u_\varepsilon(x) \sim u(x) + \varepsilon v\left(x, \frac{x}{\varepsilon}\right)$ ”

\uparrow \uparrow
 \mathbb{R}^d network !

TH 1 (micro, corrector) $\exists G(p_1, \dots, p_k) \nearrow$ in $p_i \forall i$ such that $\exists v$
 (unique up to an additive constant) sol. of

$$\left\{ \begin{array}{l} \bar{F}(x, D^2 v) = 0 \text{ in } \Sigma_1, \quad \frac{\partial v}{\partial n} = 0 \text{ on } \partial \Sigma_1 \\ \lim_{x_i \rightarrow +\infty} \frac{\sigma(x_i, y_i)}{x_i} = p_i \text{ unifly in } y_i \end{array} \right.$$

IF AND ONLY IF

$$G(p_1, \dots, p_k) = 0$$

RK: electrostatics $-c_i \Delta u_\varepsilon$ in each branch

then

$$G = \sum_{i=1}^k c_i \text{ (area } \Gamma^i) p_i$$

IF $c_i \equiv 1$, area (Γ^i) indt of i ,

THEN THE ABOVE CONDITION READS

$$\sum_{i=1}^K p_i = 0 \quad (K. \text{ law!})$$

TH 2 (ASYMPTOTIC ANALYSIS)

u_ε converges uniformly to the unique solution u of the junction problem with the nonlinear K -condition

$$G\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_\alpha}\right) = 0.$$

III. HJ (+) CASE

III.1 SETTING

EX.
$$H_1 = H_1 \left(x, \frac{\partial u}{\partial x} \right) \quad x < 0$$
$$H_2 = H_2 \left(y, \frac{\partial u}{\partial y} \right) \quad y < 0$$
$$H_3 = H_3 \left(z, \frac{\partial u}{\partial z} \right) \quad z < 0$$

$$\Sigma = \{x \leq 0, y = z = 0\} \cup \{y \leq 0, x = z = 0\} \cup \{z \leq 0, x = y = 0\}.$$

u continuous on $\Sigma (\in \mathbb{R}^3)$, same problem for any $K \geq 2$

$$\begin{cases} H_i(q, p) \rightarrow +\infty \text{ as } |p| \rightarrow \infty \text{ (uniformly in } q \leq 0) \\ H_i(q, p) \text{ continuous in } q \text{ (unifly for } p \text{ bded)} \end{cases}$$

Obviously, non unique sol's parametrized by $c = u(0)$

TH 1 : $\exists \bar{c} < \infty, \forall c \in] - \infty, \bar{c}], \exists! u$ solution s.t. $u(0) = c$

(nontrivial in the non convex case)

III.2 QUESTIONS

i) \bar{c} maximal solution ?

ii) viscosity: $-\varepsilon u_{xx}, -\varepsilon u_{yy}, -\varepsilon u_{zz}$.

ex: $\exists! u_\varepsilon$ s.t. $\frac{\partial u_\varepsilon}{\partial x} + \frac{\partial u_\varepsilon}{\partial y} + \frac{\partial u_\varepsilon}{\partial z} = 0$ (or nonlinear)

$$u_\varepsilon \xrightarrow{?} \varepsilon \rightarrow 0$$

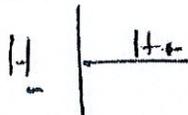
iii) discretization via monotone schemes ...

iv) asymptotic multiscale limit

v) time-dept pbs

vi) multi D junctions

vii) stratification



...

III.3 A FEW SAMPLES

TH 2 (Maximal solutions) The solution u s.t. $u(0) = \bar{c}$ (i.e. the maximal solution since any solution $\leq u(0) \dots$) is characterized by: $\forall \varphi \in C^1(\Sigma)$, if $u - \varphi$ has a maximum at 0.

$$\max_i \left\{ H_i \left(0, \frac{\partial \varphi}{\partial x_i}(0) \right) \right\} + u(0) \geq 0.$$

TH 3 (vanishing viscosity) $u_\varepsilon \rightarrow u$ uniformly as ε goes to 0 where u is the unique viscosity solution of

$$u + H_i \left(x_i, \frac{\partial u}{\partial x_i} \right) = 0 \text{ for } x_i < 0, \text{ for all } 1 \leq i \leq K$$

$$\sum_{i=1}^K \frac{\partial u}{\partial x_i} = 0 \text{ at } 0$$

RK

where the $K - BC$ is understood in viscosity sense i.e., for instance for subsolutions,

$$\left\{ \begin{array}{l} \text{for any } \varphi(x_1, \dots, x_K) \in C^1 \left(C_{i=1}^K \cup \{x_i \leq 0\} \right), \text{ if} \\ u - \varphi \text{ has a maximum over } \Sigma \text{ at } 0, \text{ then} \\ \min \left(\sum_{i=1}^K \frac{\partial \varphi}{\partial x_i} (0), \min_{1 \leq i \leq K} \left(H_i \left(0, \frac{\partial \varphi}{\partial x_i} (0) \right) \right) + u(0) \right) \leq 0 \end{array} \right.$$