NONLINEAR KORN INEQUALITIES ON A SURFACE AND APPLICATIONS

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1. THE FUNDAMENTAL FORMS OF A SURFACE

$\alpha, \beta, \ldots \in \{1, 2\}$

Summation convention

ω: open in $\mathbb{R}^2$

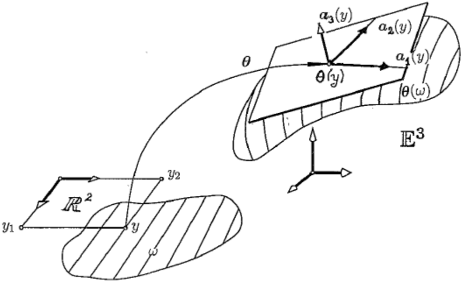
$\boldsymbol{\theta} : \omega \subset \mathbb{R}^2 \to \boldsymbol{\theta}(\omega) \subset \mathbb{E}^3$

$\boldsymbol{\theta}(\omega)$: **surface**

y_1, y_2: **curvilinear coordinates**

$\partial_\alpha := \partial/\partial y_\alpha$,
$\partial_{\alpha\beta} := \partial^2/\partial y_\alpha \partial y_\beta$, etc.

$\mathbb{E}^3$: 3d-Euclidean space



$\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbb{E}^3)$ is an **immersion**: at each $y \in \omega$,
$\boldsymbol{a}_\alpha(y) \stackrel{\text{def}}{=} \partial_\alpha \boldsymbol{\theta}(y)$ are *linearly independent*

$\mathbb{S}^2 \stackrel{\text{def}}{=} \{2 \times 2 \text{ symmetric matrices}\}$

$\mathbb{S}^2_> \stackrel{\text{def}}{=} \{\boldsymbol{A} \in \mathbb{S}^2; \boldsymbol{A} \text{ is positive definite}\}$

Covariant basis: $\boldsymbol{a}_\alpha \stackrel{\text{def}}{=} \partial_\alpha \boldsymbol{\theta}, \quad \boldsymbol{a}_3 \stackrel{\text{def}}{=} \dfrac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|}$: vector fields defined over ω

First fundamental form: $a_{\alpha\beta} \stackrel{\text{def}}{=} \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta}$

Second fundamental form: $b_{\alpha\beta} \stackrel{\text{def}}{=} \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}_3 = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \dfrac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|}$

First fundamental form: "*metric notions*", such as lengths, areas, angles $\therefore$ a.k.a. **metric tensor**

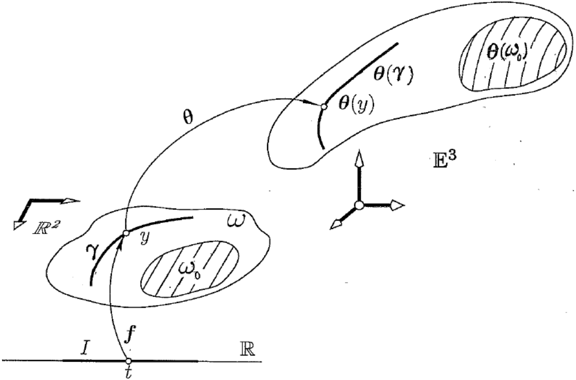
$(a_{\alpha\beta}) \colon \omega \to \mathbb{S}^2_>$: symmetric positive-definite matrix field defined over ω

Second fundamental form: "*curvature notions*"

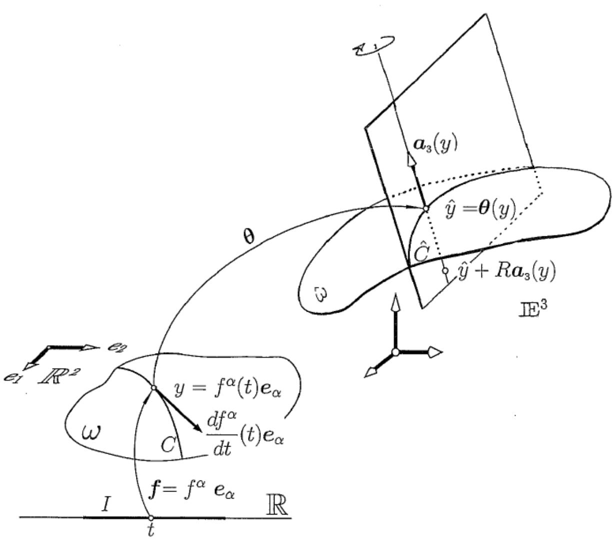
$(b_{\alpha\beta}) \colon \omega \to \mathbb{S}^2$: *symmetric* matrix field defined over ω

Remark. $\boldsymbol{a}_\beta \cdot \boldsymbol{a}_3 = 0$ implies $b_{\alpha\beta} = \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}_3 = -\boldsymbol{a}_\beta \cdot \partial_\alpha \boldsymbol{a}_3$.

$$\textbf{length} \text{ of } \boldsymbol{\theta}(\boldsymbol{\gamma}) = \int_I \sqrt{a_{\alpha\beta}(\boldsymbol{f}(t))\frac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\frac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)}\,\mathrm{d}t$$



$$\text{area}\,\boldsymbol{\theta}(\omega_0) = \int_{\omega_0} \sqrt{\det(a_{\alpha\beta}(y))}\,\mathrm{d}y$$



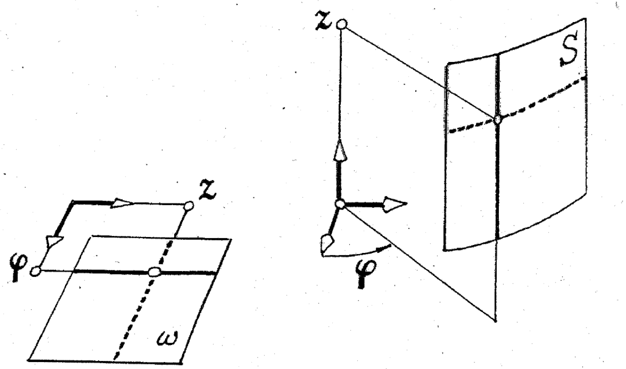
$\boldsymbol{a}_3(y)$
$\hat{y} = \boldsymbol{\theta}(y)$
$\boldsymbol{\theta}$
$\hat{C}$
$\hat{y} + R\boldsymbol{a}_3(y)$
$\hat{\omega}$
$\mathbb{E}^3$
e_2
e_1
$\mathbb{R}^2$
$y = f^\alpha(t)\boldsymbol{e}_\alpha$
ω
$\frac{df^\alpha}{dt}(t)\boldsymbol{e}_\alpha$
C
$\boldsymbol{f} = f^\alpha\ \boldsymbol{e}_\alpha$
I
$\mathbb{R}$
t

Curvature of $\boldsymbol{\theta}(\boldsymbol{\gamma})$ at $\boldsymbol{\theta}(y)$, $y = \boldsymbol{f}(t)$, when $\boldsymbol{\theta}(\boldsymbol{\gamma})$ lies in a plane normal to the surface $\boldsymbol{\theta}(\omega)$ at $\boldsymbol{\theta}(y)$:

$$\frac{1}{R} = \frac{b_{\alpha\beta}(\boldsymbol{f}(t))\dfrac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\dfrac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)}{a_{\alpha\beta}(\boldsymbol{f}(t))\dfrac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\dfrac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)}$$

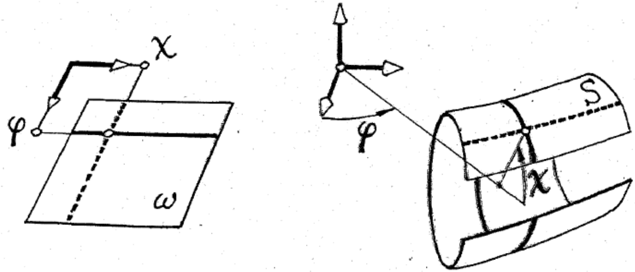
When the normal plane rotates around $\boldsymbol{a}_3(y)$, the curvature $\dfrac{1}{R}$ varies in an interval $\left[\dfrac{1}{R_1(y)}, \dfrac{1}{R_2(y)}\right]$, where $R_1(y)$ and $R_2(y)$ are the (signed) **principal radii of curvature** at $\boldsymbol{\theta}(y)$.

Portion of a **cylinder**



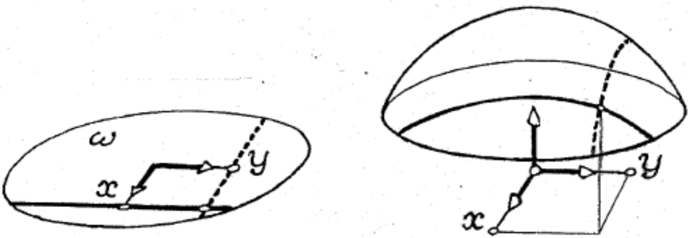
$$\boldsymbol{\theta} : (\varphi, z) \to \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ z \end{pmatrix}$$

Portion of a **torus**



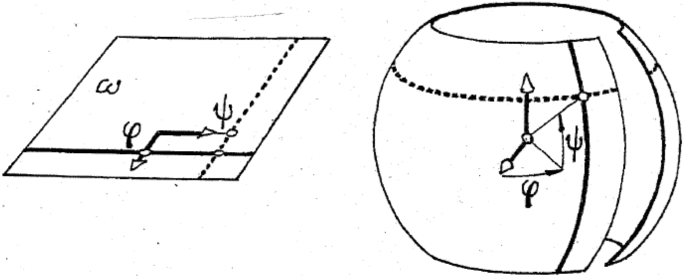
$$\boldsymbol{\theta} : (\varphi, \chi) \to \begin{pmatrix} (R + r\cos\chi)\cos\varphi \\ (R + r\cos\chi)\sin\varphi \\ r\sin\chi \end{pmatrix}$$

Portion of a **sphere**
Cartesian coordinates



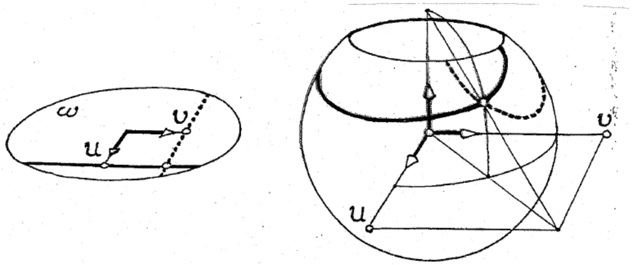
$$\boldsymbol{\theta} : (x, y) \to \begin{pmatrix} x \\ y \\ \sqrt{R^2 - (x^2 + y^2)} \end{pmatrix}$$

Spherical coordinates



$$\boldsymbol{\theta} : (\varphi, \psi) \to \begin{pmatrix} R\cos\psi\cos\varphi \\ R\cos\psi\sin\varphi \\ R\sin\psi \end{pmatrix}$$

Stereographic coordinates



$$\boldsymbol{\theta} : (u, v) \to \frac{1}{(u^2 + v^2 + R^2)} \begin{pmatrix} 2R^2 u \\ 2R^2 v \\ R(u^2 + v^2 - R^2) \end{pmatrix}$$

Let $(a^{\alpha\beta}) \stackrel{\text{def}}{=} (a_{\alpha\beta})^{-1}$ and $b_\alpha^\beta \stackrel{\text{def}}{=} a^{\sigma\beta} b_{\alpha\sigma}$ (α: row index)

$$\boxed{\begin{aligned} H &\stackrel{\text{def}}{=} \frac{1}{2}\operatorname{tr}(b_\alpha^\beta) = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) : \ \textbf{mean curvature}, \\ K &\stackrel{\text{def}}{=} \det(b_\alpha^\beta) = \frac{1}{R_1 R_2} : \ \textbf{Gaussian curvature} \end{aligned}}$$

$$\boxed{\textbf{Third fundamental form}: \ c_{\alpha\beta} \stackrel{\text{def}}{=} \partial_\alpha \boldsymbol{a}_3 \cdot \partial_\beta \boldsymbol{a}_3}$$

$(c_{\alpha\beta})$: *symmetric nonnegative-definite* matrix field defined over ω; may be viewed as the metric tensor of the *Gauß map*:

$$y \in \omega \to \boldsymbol{a}_3(y) \in \{\text{unit sphere in } \mathbb{E}^3\}$$

$$\boxed{\begin{aligned} c_{\alpha\beta} &= 2Hb_{\alpha\beta} - Ka_{\alpha\beta} \text{ in } \omega \\ c_{\alpha\beta} &= b_{\alpha\sigma} a^{\sigma\tau} b_{\tau\beta} = b_{\alpha\sigma} b_\beta^\sigma \text{ in } \omega \end{aligned}}$$

2. RECOVERY OF A SURFACE FROM ITS FUNDAMENTAL FORMS IN THE SPACE $\mathcal{C}^3(\omega)$

Preliminary observation: Let an open set $\omega \subset \mathbb{R}^2$ and an immersion $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}^3(\omega; \mathbb{E}^3)$ be given. Then the components $a_{\alpha\beta} \in \mathcal{C}^2(\omega)$ and $b_{\alpha\beta} \in \mathcal{C}^1(\omega)$ of the two fundamental forms of the surface $\boldsymbol{\theta}(\omega)$ *cannot be arbitrary functions*: Let

$$\boxed{(a^{\sigma\tau}) \stackrel{\text{def}}{=} (a_{\alpha\beta})^{-1}, \quad \Gamma_{\alpha\beta\tau} \stackrel{\text{def}}{=} \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}_\tau \quad \text{and} \quad \Gamma^\sigma_{\alpha\beta} \stackrel{\text{def}}{=} a^{\sigma\tau}\Gamma_{\alpha\beta\tau}}$$

The functions $\Gamma_{\alpha\beta\tau} \in \mathcal{C}^1(\omega)$ and $\Gamma^\sigma_{\alpha\beta} \in \mathcal{C}^1(\omega)$ are the **Christoffel symbols**. Then it is easy to see that:

$$\begin{aligned}
\partial_{\alpha\sigma}\boldsymbol{a}_\beta \cdot \boldsymbol{a}_\tau &= \partial_\sigma \Gamma_{\alpha\beta\tau} - \Gamma^\mu_{\alpha\beta}\Gamma_{\sigma\tau\mu} - b_{\alpha\beta}b_{\sigma\tau}, \\
\partial_{\alpha\sigma}\boldsymbol{a}_\beta \cdot \boldsymbol{a}_3 &= \partial_\sigma b_{\alpha\beta} + \Gamma^\mu_{\alpha\beta} b_{\sigma\mu}.
\end{aligned}$$

Besides,

$$\boxed{\partial_{\alpha\sigma\beta}\boldsymbol{\theta} = \partial_{\alpha\beta\sigma}\boldsymbol{\theta}} \iff \partial_{\alpha\sigma}\boldsymbol{a}_\beta = \partial_{\alpha\beta}\boldsymbol{a}_\sigma \iff \begin{cases} \partial_{\alpha\sigma}\boldsymbol{a}_\beta \cdot \boldsymbol{a}_\tau = \partial_{\alpha\beta}\boldsymbol{a}_\sigma \cdot \boldsymbol{a}_\tau \\ \partial_{\alpha\sigma}\boldsymbol{a}_\beta \cdot \boldsymbol{a}_3 = \partial_{\alpha\beta}\boldsymbol{a}_\sigma \cdot \boldsymbol{a}_3 \end{cases}$$

Consequently, the following **necessary conditions** hold:

$$\partial_\beta \Gamma_{\alpha\sigma\tau} - \partial_\sigma \Gamma_{\alpha\beta\tau} + \Gamma^\mu_{\alpha\beta}\Gamma_{\sigma\tau\mu} - \Gamma^\mu_{\alpha\sigma}\Gamma_{\beta\tau\mu} = b_{\alpha\sigma}b_{\beta\tau} - b_{\alpha\beta}b_{\sigma\tau} \text{ in } \omega$$

Gauß equations

$$\partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + \Gamma^\mu_{\alpha\sigma} b_{\beta\mu} - \Gamma^\mu_{\alpha\beta} b_{\sigma\mu} = 0 \text{ in } \omega$$

Codazzi-Mainardi equations

It is easy to see that the Christoffel symbols $\Gamma_{\alpha\beta\tau} = \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}_\tau$ and $\Gamma^\sigma_{\alpha\beta} = a^{\sigma\tau}\Gamma_{\alpha\beta\tau}$ can be expressed *solely in terms of the components of the first fundamental form*:

$$\Gamma_{\alpha\beta\tau} = \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}) \text{ and } \Gamma^\sigma_{\alpha\beta} = a^{\sigma\tau}\Gamma_{\alpha\beta\tau}$$
$$\text{with } (a^{\sigma\tau}) = (a_{\alpha\beta})^{-1}$$

Consequently, *the Gauß and Codazzi-Mainardi equations are (nonlinear) compatibility relations that are necessarily satisfied by the first and second fundamental forms*.

It is a *fundamental* result that, if ω is *simply-connected*, these compatibility conditions become also *sufficient* for the existence of a surface $\boldsymbol{\theta}(\omega)$ with $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ as its fundamental forms.

$\mathbb{O}^3_+ \stackrel{\text{def}}{=} \{$ proper orthogonal 3×3 matrices $\}$

FUNDAMENTAL THEOREM OF SURFACE THEORY IN THE SPACE $\mathcal{C}^3(\omega)$: $\omega \subset \mathbb{R}^2$: *open and simply-connected. Let there be given* $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>)$ *and* $(b_{\alpha\beta}) \in \mathcal{C}^1(\omega; \mathbb{S}^2)$ *that satisfy the* **Gauß and Codazzi-Mainardi equations** *in* ω. *Then there exists an immersion* $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbb{E}^3)$ *such that:*

$$\boxed{\partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} = a_{\alpha\beta} \text{ and } \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} = b_{\alpha\beta} \text{ in } \omega}$$

Uniqueness holds up to **isometric equivalence** *in* $\mathbb{E}^3$: *All other solutions* $\boldsymbol{\theta}^\sharp \in \mathcal{C}^3(\omega; \mathbb{E}^3)$ *are given by:*

$$\boxed{\boldsymbol{\theta}^\sharp(y) = \boldsymbol{a} + \boldsymbol{Q}\boldsymbol{\theta}(y), y \in \omega, \text{ with } \boldsymbol{a} \in \mathbb{E}^3 \text{ and } \boldsymbol{Q} \in \mathbb{O}^3_+ \Longleftrightarrow (\boldsymbol{\theta}^\sharp, \boldsymbol{\theta}) \in \mathcal{R}}$$

Uniqueness holds if $\boldsymbol{\theta}$ *is subjected to Cauchy conditions of the form*

$$\boxed{\boldsymbol{\theta}(y_0) = \boldsymbol{\theta}_0 \text{ and } \partial_\alpha \boldsymbol{\theta}(y_0) = \boldsymbol{a}^0_\alpha,}$$

with $y_0 \in \omega, \boldsymbol{\theta}_0 \in \mathbb{E}^3$, *and* $\boldsymbol{a}^0_\alpha \in \mathbb{E}^3$ *such that* $\boldsymbol{a}^0_\alpha \cdot \boldsymbol{a}^0_\beta = \boldsymbol{a}^0_{\alpha\beta}(y_0)$. $\square$

Brief outline of the proof: (a) *Preliminary observation.* Let an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbb{E}^3)$ be *given*. Then $\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$ and $\boldsymbol{a}_3 = \dfrac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|}$ satisfy:

equations of Gauß: $\partial_\alpha \boldsymbol{a}_\beta = \Gamma^\sigma_{\alpha\beta} \boldsymbol{a}_\sigma + b_{\alpha\beta} \boldsymbol{a}_3$ in ω

equations of Weingarten: $\partial_\alpha \boldsymbol{a}_3 = -a^{\beta\sigma} b_{\alpha\sigma} \boldsymbol{a}_\beta$ in ω

$\mathbb{M}^3 = \{3 \times 3 \text{ matrices}\}$

The equations of Gauß and Weingarten can be re-written as a single matrix equation as

$$\partial_\alpha \boldsymbol{F} = \boldsymbol{F} \boldsymbol{\Gamma}_\alpha \text{ with } \boldsymbol{F} \stackrel{\text{def}}{=} \left(\boldsymbol{a}_1 \middle| \boldsymbol{a}_2 \middle| \boldsymbol{a}_3\right) \text{ and } \boldsymbol{\Gamma}_\alpha \stackrel{\text{def}}{=} \begin{pmatrix} \Gamma^1_{\alpha 1} & \Gamma^1_{\alpha 2} & -a^{1\sigma} b_{\alpha\sigma} \\ \Gamma^2_{\alpha 1} & \Gamma^2_{\alpha 2} & -a^{2\sigma} b_{\alpha\sigma} \\ b_{\alpha_1} & b_{\alpha 2} & 0 \end{pmatrix}$$

clearly, $\boldsymbol{F} \in \mathcal{C}^2(\omega; \mathbb{M}^3)$ and $\boldsymbol{\Gamma}_\alpha \in \mathcal{C}^1(\omega; \mathbb{M}^3)$

(b) *Given* $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in \mathcal{C}^1(\omega; \mathbb{S}^2)$ satisfying the Gauß and Codazzi-Mainardi equations, it is thus natural to seek a solution $\boldsymbol{F} \in \mathcal{C}^2(\omega; \mathbb{M}^3)$ to:

$$\boxed{\textbf{Pfaff system: } \partial_\alpha \boldsymbol{F} = \boldsymbol{F}\boldsymbol{\Gamma}_\alpha \text{ in } \omega}$$

Such a Pfaff system has a solution if ω is *simply-connected* and the matrix fields $\boldsymbol{\Gamma}_\alpha$ satisfy

$$\boxed{\textbf{compatibility conditions: } \partial_\alpha \boldsymbol{\Gamma}_\beta - \partial_\beta \boldsymbol{\Gamma}_\alpha + \boldsymbol{\Gamma}_\alpha \boldsymbol{\Gamma}_\beta - \boldsymbol{\Gamma}_\beta \boldsymbol{\Gamma}_\alpha = \boldsymbol{0} \text{ in } \omega,}$$

which are precisely *equivalent* to the Gauß and Codazzi-Mainardi equations!

Remark. If $\boldsymbol{F}$ is invertible in ω, these compatibility conditions are clearly *necessary*, since they simply express that $\partial_{\alpha\beta}\boldsymbol{F} = \partial_{\beta\alpha}\boldsymbol{F}$ in ω.

Pfaff systems: See, e.g., Sect. 6.20 in
P.G. Ciarlet: *Linear and Nonlinear Functional Analysis with Applications*, SIAM (2013).

(c) Given $\boldsymbol{F} = \left(\boldsymbol{a}_1 \middle| \boldsymbol{a}_2 \middle| \boldsymbol{a}_3\right) \in \mathcal{C}^2(\omega; \mathbb{M}^3)$ found as in (b), it is then natural to seek a solution $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbb{E}^3)$ to:

$$\boxed{\textbf{Poincaré system: } \partial_\alpha \boldsymbol{\theta} = \boldsymbol{a}_\alpha \text{ in } \omega.}$$

Such a Poincaré system has a solution if ω is *simply-connected* and the vector fields $\boldsymbol{a}_\alpha$ satisfy

$$\boxed{\textbf{compatibility conditions: } \partial_\alpha \boldsymbol{a}_\beta = \partial_\beta \boldsymbol{a}_\alpha \text{ in } \omega,}$$

which are equivalent to the *symmetries*

$$\Gamma^\sigma_{\alpha\beta} = \Gamma^\sigma_{\beta\alpha} \text{ and } b_{\alpha\beta} = b_{\beta\alpha}.$$

Poincaré systems: See, e.g., Sect. 6.17 in ibid.

Remark. These compatibility conditions are clearly *necessary*, since they simply express that $\partial_{\alpha\beta}\boldsymbol{\theta} = \partial_{\beta\alpha}\boldsymbol{\theta}$ in ω.

(d) One next shows that

$$(*) \qquad \boxed{\partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} = a_{\alpha\beta} \text{ and } \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} = b_{\alpha\beta} \text{ in } \omega}$$

Proof relies on *uniqueness of solutions to the Cauchy problem for Pfaff systems* (only connectedness of ω is used).

(e) One finally shows that, if $\boldsymbol{\theta}$ and $\boldsymbol{\theta}^\sharp$ are two solutions to $(*)$, then there exist $\boldsymbol{a} \in \mathbb{E}^3$ and $\boldsymbol{Q} \in \mathbb{O}^3_+$ such that

$$\boxed{\boldsymbol{\theta}^\sharp(y) = \boldsymbol{a} + \boldsymbol{Q}\boldsymbol{\theta}(y),\ y \in \omega} \quad \Longleftrightarrow \quad \boxed{(\boldsymbol{\theta}^\sharp, \boldsymbol{\theta}) \in \mathcal{R}}$$

Proof relies again on *uniqueness of solutions to the Cauchy problem for Pfaff systems* and on the *implication* $\partial_\alpha \boldsymbol{\theta} = \mathbf{0}$ in $\omega \Rightarrow \boldsymbol{\theta}$ is constant (only connectedness of ω is used). $\square$

Remark. A similar result holds if $(a_{\alpha\beta}) \in \mathcal{C}^1(\omega; \mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in \mathcal{C}^0(\omega; \mathbb{S}^2)$:
P. Hartman, A. Wintner: *American J. Math* (1950).

3. SOME QUESTIONS AND APPLICATIONS

COROLLARY: *There exists a well-defined mapping:*

$$\boxed{\boldsymbol{F} : \left\{ \begin{array}{ll} (a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>) & \text{satisfying the } \textbf{Gauß} \text{ and} \\ (b_{\alpha\beta}) \in \mathcal{C}^1(\omega; \mathbb{S}^2) & \textbf{Codazzi-Mainardi} \text{ eqs.} \end{array} \right\} \to \dot{\boldsymbol{\theta}} \in \mathcal{C}^3(\omega; \mathbb{E}^3)/\mathcal{R}}$$

$\square$

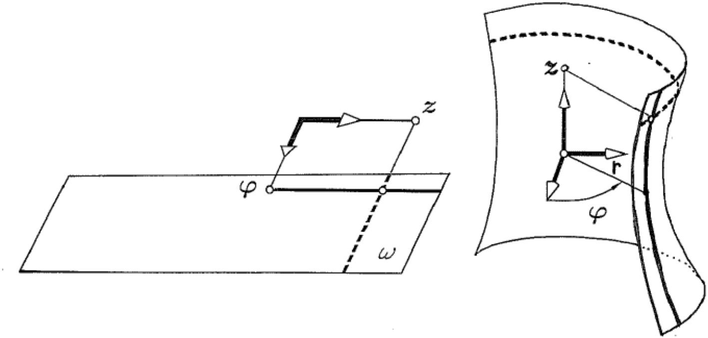
Questions:

1. Is the *mapping* $\boldsymbol{F}$ *continuous* when the spaces $\mathcal{C}^m(\omega)$ are equipped with their natural *Fréchet topology*?
2. Can the *fundamental theorem of surface theory* be extended to *other function spaces*, such as $\mathcal{C}^m(\overline{\omega})$ or $W^{m,p}(\omega)$?
3. If so, are the corresponding mappings $\boldsymbol{F}$ *Lipschitz-continuous*? Equivalently, do they satisfy *nonlinear Korn inequalities on surfaces*?

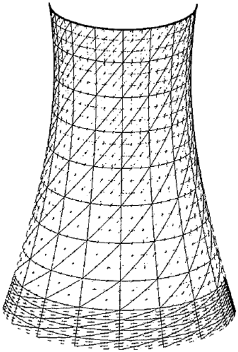
Applications:

1. *Differential geometry of surfaces in* $\mathbb{R}^3$ per se
2. *Intrinsic nonlinear shell theory*
3. *Modelling of the Earth surface*

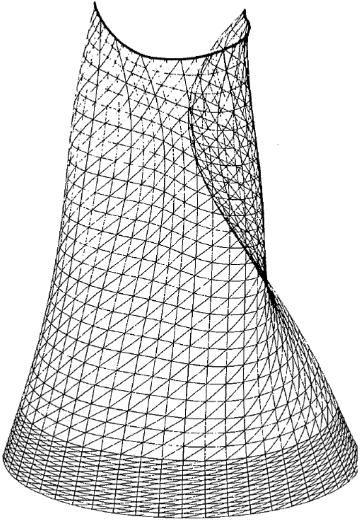
Portion of a **hyperboloid of revolution**



$$\boldsymbol{\theta} : (\varphi, z) \to \begin{pmatrix} r\sqrt{1+z^2}\cos\varphi \\ r\sqrt{1+z^2}\sin\varphi \\ z \end{pmatrix}$$



Cooling tower: Triangulation.
Reproduced by courtesy of Professor J.H. Argyris.



Cooling tower: Deformed structure under wind load.
Reproduced by courtesy of Professor J.H. Argyris.

In the **intrinsic approach to nonlinear shell theory**, one computes directly the *two fundamental forms of the deformed middle surface* of the shell, instead of the displacement vector field as in the classical approach:
S. Opoka & W. Pietraszkiewicz: *Intern. J. Solids Structures* (2004)
W. Pietraszkiewicz & M.L. Szwabowicz: *Intern. J. Solids Structures* (2007)

Mathematical analysis of the intrinsic approach: *Linearly elastic shells:*
P.G. Ciarlet & L. Gratie: *Math. Models Methods Appl. Sci*. (2005)
C. Amrouche, P.G. Ciarlet, L. Gratie & S. Kesavan: *J. Math. Pures Appl*. (2006)
P.G. Ciarlet, L. Gratie, C. Mardare, Shen Ming: *Math. Models Methods Appl. Sci*. (2008)
P.G. Ciarlet, O. Iosifescu: *J. Math. Pures Appl*. (2014)
P.G. Ciarlet, C. Mardare & Shen Xiaoqin: *Anal. Appl*. (2015)
Nonlinearly elastic shells: Virgin territory!
Numerical analysis: Virgin territory! (even for linearly elastic shells)

Modelling of the Earth surface. New numerical methods have been recently introduced for reconstructing the Earth surface from the knowledge of its *two fundamental forms*:

"HASM" = "High-Accuracy Surface Modelling"

"DEM" = "Digital Elevation Model"

Tian-Xiang Yue, Dun-Jiang Song, Zhen-Ping Du & Wei Wang: *Intern. J. Remote Sensing* (2010)

Numerical evidence seems to indicate that such methods perform better than the classical methods for modelling the Earth surface, using, e.g., splines (see Fig. 4 in ibid.).

4. CONTINUITY OF A SURFACE AS A FUNCTION OF ITS FUNDAMENTAL FORMS IN THE SPACE $\mathcal{C}^3(\omega)$

$|\cdot| \stackrel{\text{def}}{=}$ Euclidean vector norm in $\mathbb{R}^N$ and spectral norm of matrices
$\kappa \Subset \omega$ means that κ is a compact subset of ω
Given $f \in \mathcal{C}^\ell(\omega;\mathbb{R})$ or $\boldsymbol{\theta} \in \mathcal{C}^\ell(\omega;\mathbb{R}^d)$,

$$\|f\|_{\ell,\kappa} \stackrel{\text{def}}{=} \sup_{\left\{ \substack{y\in\kappa \\ |\boldsymbol{\alpha}|\le\ell} \right.} |\partial^{\boldsymbol{\alpha}} f(y)| \quad \|\boldsymbol{\theta}\|_{\ell,\kappa} \stackrel{\text{def}}{=} \sup_{\left\{ \substack{y\in\kappa \\ |\boldsymbol{\alpha}|\le\ell} \right.} |\partial^{\boldsymbol{\alpha}} \boldsymbol{\theta}(y)|$$

(multi-index notation $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ with $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2$ is used)

Then, equipped with the semi-norms $\|\cdot\|_{\ell,\kappa}$ for all $\kappa \Subset \omega$, the space $\mathcal{C}^\ell(\omega;\mathbb{R}^d)$ becomes a locally convex topological space. The corresponding topology, which is metrizable but not normable, is called **Fréchet topology**. In this topology,

$$\boxed{\psi^k \underset{k\to\infty}{\longrightarrow} \psi \text{ as } k \to \infty \iff \forall \kappa \Subset \omega,\ \|\psi^k - \psi\|_{\ell,\kappa} \underset{k\to\infty}{\longrightarrow} 0}$$

Equivalence class $\dot{\boldsymbol{\theta}}$ *of* $\boldsymbol{\theta} : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ *modulo* $\mathcal{R}$:

$$\boxed{\begin{aligned}\dot{\boldsymbol{\theta}} = \{\boldsymbol{\theta}^\sharp : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3 \text{ with } \boldsymbol{\theta}^\sharp(y) = \boldsymbol{a} + \boldsymbol{Q}\boldsymbol{\theta}(y),\ y \in \omega, \\ \text{for some } \boldsymbol{a} \in \mathbb{E}^3,\ \boldsymbol{Q} \in \mathbb{O}^3_+\}\end{aligned}}$$

The relation $\mathcal{R}$ is called **isometric equivalence**.

THEOREM: $\omega \subset \mathbb{R}^2$: *open, simply-connected. Given immersions* $\boldsymbol{\theta}^k \in \mathcal{C}^3(\omega; \mathbb{E}^3)$, $k \geq 1$, *and* $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbb{E}^3)$*, let*

$$a^k_{\alpha\beta} := \partial_\alpha \boldsymbol{\theta}^k \cdot \partial_\beta \boldsymbol{\theta}^k \quad \text{and} \quad b^k_{\alpha\beta} := \partial_{\alpha\beta}\boldsymbol{\theta}^k \cdot \frac{\partial_1 \boldsymbol{\theta}^k \wedge \partial_2 \boldsymbol{\theta}^k}{|\partial_1 \boldsymbol{\theta}^k \wedge \partial_2 \boldsymbol{\theta}^k|} \text{ in } \omega,$$

$$a_{\alpha\beta} := \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \quad \text{and} \quad b_{\alpha\beta} := \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \text{ in } \omega$$

Assume that

$$\boxed{\forall \kappa \Subset \omega,\ \|a^k_{\alpha\beta} - a_{\alpha\beta}\|_{2,\kappa} \underset{k\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \|b^k_{\alpha\beta} - b_{\alpha\beta}\|_{1,\kappa} \underset{k\to\infty}{\longrightarrow} 0}$$

Then there exist $\boldsymbol{\theta}^{\sharp k} \in \dot{\boldsymbol{\theta}}^k$, $k \geq 1$, *such that*

$$\boxed{\forall \kappa \Subset \omega,\ \|\boldsymbol{\theta}^{\sharp k} - \boldsymbol{\theta}\|_{3,\kappa} \underset{k\to\infty}{\longrightarrow} 0}$$

□

This result implies the following *continuity result*:

COROLLARY. $\omega \subset \mathbb{R}^2$: *open, simply-connected. Let the spaces* $\mathcal{C}^\ell(\omega;\mathbb{S}^2)$, $\ell = 1, 2$, *and* $\mathcal{C}^3(\omega;\mathbb{E}^3)$ *be equipped with their Fréchet topologies. Then the following mapping is continuous:*

$$\boxed{\boldsymbol{F} : \left\{ \begin{array}{ll} (a_{\alpha\beta}) \in \mathcal{C}^2(\omega;\mathbb{S}^2_>) & \text{satisfying the } \textbf{Gauß} \text{ and} \\ (b_{\alpha\beta}) \in \mathcal{C}^1(\omega;\mathbb{S}^2) & \textbf{Codazzi-Mainardi} \text{ eqs.} \end{array} \right\} \to \dot{\boldsymbol{\theta}} \in \mathcal{C}^3(\omega;\mathbb{E}^3)/\mathcal{R}}$$

P.G. Ciarlet: *J. Math. Pures Appl.* (2003)

Proof relies on an analogous result "in 3d": A 3d-deformation is a continuous function of its metric tensor:
P.G. Ciarlet & F. Laurent: *Arch. Rational Mech. Anal.* (2003)

5. RECOVERY OF A SURFACE FROM ITS FUNDAMENTAL FORMS IN THE SPACE $\mathcal{C}^3(\overline{\omega})$

FUNDAMENTAL THEOREM OF SURFACE THEORY REVISITED, IN THE SPACE $\mathcal{C}^3(\overline{\omega})$: $\omega \subset \mathbb{R}^2$: *open, bounded, simply-connected; Lipschitz-continuous boundary. Given* $(a_{\alpha\beta}) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^2_>)$ *and* $(b_{\alpha\beta}) \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2)$ *that satisfy the Gauß and Codazzi-Mainardi equations in* ω, *there exists* $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ *such that*:

$$\boxed{a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \quad \text{and} \quad b_{\alpha\beta} = \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \text{ in } \overline{\omega}}$$

Uniqueness holds up to isometric equivalence in $\mathbb{E}^3$: *All other solutions* $\boldsymbol{\theta}^\sharp$ *are given by*

$$\boxed{\boldsymbol{\theta}^\sharp(y) = \boldsymbol{a} + \boldsymbol{Q}\boldsymbol{\theta}(y),\ y \in \overline{\omega}, \text{ with } \boldsymbol{a} \in \mathbb{E}^3 \text{ and } \boldsymbol{Q} \in \mathbb{O}^3_+ \iff \boldsymbol{\theta}^\sharp \in \dot{\boldsymbol{\theta}}.} \qquad \square$$

P.G. Ciarlet & C. Mardare, *Analysis and Applications* (2005)

The proof (long and technical) consists in showing that, under the above assumptions on ω, the solution $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbb{E}^3)$ obtained in the previous existence theorem can be extended, together with all its partial derivatives $\partial^{\boldsymbol{\alpha}}\boldsymbol{\theta}$, $|\boldsymbol{\alpha}| \leq 3$, to continuous functions over $\overline{\omega}$.

6. CONTINUITY OF A SURFACE AS A FUNCTION OF ITS FUNDAMENTAL FORMS IN THE SPACE $\mathcal{C}^3(\overline{\omega})$; A KORN INEQUALITY ON A SURFACE IN THE SPACE $\mathcal{C}^3(\overline{\omega})$

THEOREM: $\omega \subset \mathbb{R}^2$: *open, bounded, simply-connected; Lipschitz-continuous boundary. Let the spaces* $\mathcal{C}^\ell(\overline{\omega}; \mathbb{S}^2)$, $\ell = 1, 2$, *and* $\mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ *be equipped with their norm topologies. Then the following mapping* $\boldsymbol{F}$ *between subsets of Banach spaces is* **locally Lipschitz-continuous**:

$$\boldsymbol{F} : \left\{ \begin{array}{ll} (a_{\alpha\beta}) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^2_>) & \text{satisfying the } \textbf{Gauß} \text{ and} \\ (b_{\alpha\beta}) \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2) & \textbf{Codazzi-Mainardi} \text{ equations in } \omega \end{array} \right\} \to \dot{\boldsymbol{\theta}} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)/\mathcal{R}$$

Equivalently, the following **nonlinear Korn's inequality on a surface holds**: *Given any immersion* $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ *with fundamental forms* $(a_{\alpha\beta}) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^2_>)$ *and* $(b_{\alpha\beta}) \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2)$, *there exist a constant* $c(\boldsymbol{\theta})$ *and a neighborhood* $\mathcal{V}$ *of* $((a_{\alpha\beta}), (b_{\alpha\beta}))$ *in* $\mathcal{C}^2(\overline{\omega}, \mathbb{S}^2_>) \times \mathcal{C}^1(\overline{\omega}, \mathbb{S}^2)$ *with the following property: Given any* $\tilde{\boldsymbol{\theta}}$ *such that* $((\tilde{a}_{\alpha\beta}), (\tilde{b}_{\alpha\beta})) \in \mathcal{V}$ (with self-explanatory notations), *there exists an immersion* $\tilde{\boldsymbol{\theta}}^\sharp$ *isometrically equivalent to* $\tilde{\boldsymbol{\theta}}$ *such that*

$$\boxed{\begin{aligned} &\|\tilde{\boldsymbol{\theta}}^\sharp - \boldsymbol{\theta}\|_{\mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)} \\ &\qquad \le c(\boldsymbol{\theta}) \big\{ \|(\tilde{a}_{\alpha\beta} - a_{\alpha\beta})\|_{\mathcal{C}^2(\overline{\omega}; \mathbb{S}^2)} + \|(\tilde{b}_{\alpha\beta} - b_{\alpha\beta})\|_{\mathcal{C}^1(\overline{\omega}; \mathbb{S}^2)} \big\} \end{aligned}} \quad \square$$

Proof is similar to that of an analogous result "in 3d":
P.G. Ciarlet & C. Mardare: *J. Math. Pures Appl.* (2004)

Extension to a *simply-connected Riemannian space* $\omega \subset \mathbb{R}^p$ *isometrically immersed in* $\mathbb{R}^{p+q}$:
M. Szopos: *Analysis and Applications* (2005)

7. NONLINEAR KORN INEQUALITIES ON A SURFACE IN THE SOBOLEV SPACES $W^{1,p}(\omega)$, $p > 1$

The following *implications* clearly hold if $p \geq 1$:

$$\left.\begin{array}{l} \boldsymbol{\theta} \in W^{1,2p}(\omega; \mathbb{E}^3), \quad \boldsymbol{a}_\alpha \stackrel{\text{def}}{=} \partial_\alpha \boldsymbol{\theta} \\ \boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \neq \mathbf{0} \text{ a.e. in } \omega \\ \boldsymbol{a}_3 \stackrel{\text{def}}{=} \dfrac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|} \in W^{1,2p}(\omega; \mathbb{E}^3) \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta \in L^p(\omega) \\ b_{\alpha\beta} = -\partial_\alpha \boldsymbol{a}_3 \cdot \boldsymbol{a}_\beta \in L^p(\omega) \\ c_{\alpha\beta} = \partial_\alpha \boldsymbol{a}_3 \cdot \partial_\beta \boldsymbol{a}_3 \in L^p(\omega) \end{array}\right.$$

Notations:

$$\|\boldsymbol{\theta}\|_{L^p(\omega;\mathbb{E}^3)} \stackrel{\text{def}}{=} \left\{ \int_\omega |\boldsymbol{\theta}(y)|^p \, \mathrm{d}y \right\}^{1/p} \quad \|\boldsymbol{A}\|_{L^p(\omega;\mathbb{S}^2)} \stackrel{\text{def}}{=} \left\{ \int_\omega |\boldsymbol{A}(y)|^p \, \mathrm{d}y \right\}^{1/p}$$

$$\|\boldsymbol{\theta}\|_{W^{1,p}(\omega;\mathbb{E}^3)} \stackrel{\text{def}}{=} \left\{ \int_\omega \Big(|\boldsymbol{\theta}(y)|^p + \sum_\alpha |\partial_\alpha \boldsymbol{\theta}(y)|^p \Big) \, \mathrm{d}y \right\}^{1/2}$$

The following theorems are due to:

P.G. Ciarlet, L. Gratie & C. Mardare: *J. Math. Pures Appl. (2006)*

P.G. Ciarlet, M. Malin & C. Mardare, in preparation

THEOREM: $\omega \subset \mathbb{R}^2$ *bounded, open, connected; Lipschitz-continuous boundary. Let* $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega}; \mathbb{E}^3)$*: immersion such that* $\boldsymbol{a}_3 \in \mathcal{C}^1(\overline{\omega}; \mathbb{E}^3)$.

Assume that $p > 1$. *Then, given any* $\varepsilon > 0$, *there exists a constant* $c(p, \boldsymbol{\theta}; \varepsilon)$ *with the following property:*

Given any mapping $\widetilde{\boldsymbol{\theta}} \in W^{1,2p}(\omega; \mathbb{E}^3)$ *such that* $\tilde{\boldsymbol{a}}_1 \wedge \tilde{\boldsymbol{a}}_2 \neq \boldsymbol{0}$ *a.e. in* ω, $\tilde{\boldsymbol{a}}_3 \in W^{1,2p}(\omega; \mathbb{E}^3)$, *and*

$$\boxed{|\tilde{R}_1| \geq \varepsilon \text{ and } |\tilde{R}_2| \geq \varepsilon \text{ a.e. in } \omega,}$$

there exists a mapping $\tilde{\boldsymbol{\theta}}^\sharp$ *isometrically equivalent to* $\tilde{\boldsymbol{\theta}}$ *such that the following* **nonlinear Korn inequality on a surface** *holds*:

$$\boxed{\begin{aligned} &\|\tilde{\boldsymbol{\theta}}^\sharp - \boldsymbol{\theta}\|_{W^{1,p}(\omega;\mathbb{E}^3)} + \|\tilde{\boldsymbol{a}}_3^\sharp - \boldsymbol{a}_3\|_{W^{1,p}(\omega;\mathbb{E}^3)} \\ &\quad \leq c(p, \boldsymbol{\theta}, \varepsilon)\Big\{\|(\tilde{a}_{\alpha\beta} - a_{\alpha\beta})\|_{L^p(\omega;\mathbb{S}^2)} + \|(\tilde{b}_{\alpha\beta} - b_{\alpha\beta})\|_{L^p(\omega;\mathbb{S}^2)} \\ &\qquad\qquad + \|(\tilde{c}_{\alpha\beta} - c_{\alpha\beta})\|_{L^p(\omega;\mathbb{S}^2)}\Big\} \end{aligned}}$$

□

Remark. We need to assume that $\tilde{\boldsymbol{\theta}} \in W^{1,2p}(\omega; \mathbb{E}^3)$ and $\tilde{\boldsymbol{a}}_3 \in W^{1,2p}(\omega; \mathbb{E}^3)$ so that $\tilde{a}_{\alpha\beta}, \tilde{b}_{\alpha\beta}, \tilde{c}_{\alpha\beta} \in L^p(\omega)$, even though the estimates are with respect to the norm $\|\cdot\|_{W^{1,p}(\omega;\mathbb{E}^3)}$.

The proof essentially relies on *two ideas*:

(a) A "**nonlinear 3d-Korn inequality**":

P.G. Ciarlet & C. Mardare, *J. Nonlinear Sci.* (2004).

P.G. Ciarlet & C. Mardare, *J. Math. Pures Appl.* (2015)

For related ideas, see also:

Y.G. Reshetnyak, *Siberian Math. J.* (2003)

THEOREM: $\Omega \subset \mathbb{R}^3$ *bounded, open, connected; Lipschitz-continuous boundary. Let* $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{E}^3)$ *be an immersion with* $\det \boldsymbol{\nabla}\boldsymbol{\Theta} > 0$ *in* $\overline{\Omega}$. *Assume that* $p > 1$. *Then there exists a constant* $C(p, \boldsymbol{\Theta})$ *with the following property:*

Given any mapping $\tilde{\boldsymbol{\Theta}} \in W^{1,2p}(\Omega; \mathbb{E}^3)$ *such that* $\det \boldsymbol{\nabla}\tilde{\boldsymbol{\Theta}} > 0$ *a.e. in* Ω, *there exists an immersion* $\tilde{\boldsymbol{\Theta}}^\sharp$ *isometrically equivalent to* $\tilde{\boldsymbol{\Theta}}$ *such that the following* **nonlinear 3d-Korn inequality** *holds*:

$$\boxed{\|\tilde{\boldsymbol{\Theta}}^\sharp - \boldsymbol{\Theta}\|_{W^{1,p}(\Omega;\mathbb{E}^3)} \leq C(p, \boldsymbol{\Theta}) \|\boldsymbol{\nabla}\tilde{\boldsymbol{\Theta}}^T \boldsymbol{\nabla}\tilde{\boldsymbol{\Theta}} - \boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta}\|_{L^p(\Omega;\mathbb{E}^3)}}$$

The proof of this *nonlinear 3d-Korn inequality* relies in particular on a *generalization* of the "**geometric rigidity lemma**" (which corresponds to $\boldsymbol{\Theta} = \mathbf{id}_{\overline{\Omega}}$):

$\Omega \subset \mathbb{R}^n$: bounded, open, connected; Lipschitz-continuous boundary. Then there exists a constant $\Lambda(\Omega)$ such that, for each $\tilde{\boldsymbol{\Theta}} \in H^1(\Omega;\mathbb{R}^n)$, there exists $\boldsymbol{R} = \boldsymbol{R}(\tilde{\boldsymbol{\Theta}}) \in \mathbb{O}^n_+$ such that

$$\|\boldsymbol{\nabla}\tilde{\boldsymbol{\Theta}} - \boldsymbol{R}\|_{L^2(\Omega;\mathbb{M}^n)} \leq \Lambda(\Omega) \quad \underbrace{\left\|\mathrm{dist}(\boldsymbol{\nabla}\tilde{\boldsymbol{\Theta}}, \mathbb{O}^n_+)\right\|_{L^2(\Omega)}}_{=\left\{\int_\Omega \inf_{\boldsymbol{Q}\in\mathbb{O}^n_+} \left|\boldsymbol{\nabla}\tilde{\boldsymbol{\Theta}}(x) - \boldsymbol{Q}\right|^2 \mathrm{d}x\right\}^{1/2}}$$

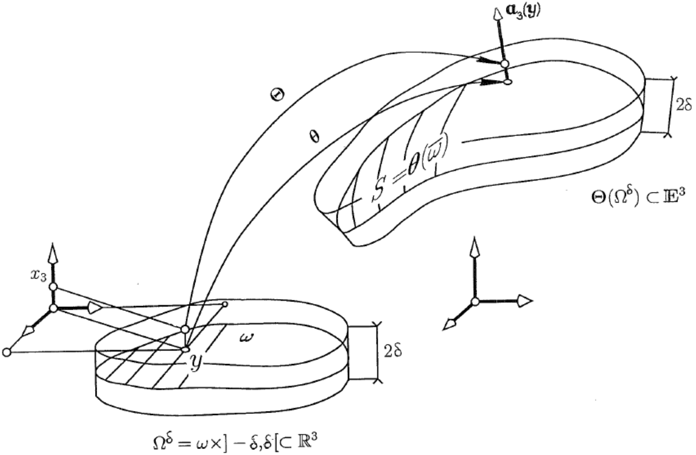
G. Friesecke, R.D. James & S. Müller, *Comm. Pure Appl. Math.* (2002).

This lemma was extended to the spaces $L^p(\Omega;\mathbb{M}^n)$, $p > 1$, in:
S. Conti, *Habilitationsschrift*, Universität Leipzig (2004).

(b) *Any smooth enough immersion* $\boldsymbol{\theta} : \omega \to \mathbb{E}^3$ *can be canonically extended to an immersion* $\boldsymbol{\Theta} : \Omega^\delta \to \mathbb{E}^3$*, with* $\Omega^\delta = \omega \times]-\delta, \delta[$ *provided* $\delta > 0$ *is small enough, with*

$$\boxed{\boldsymbol{\Theta}(y, x_3) \stackrel{\text{def}}{=} \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y) \text{ for all } (y, x_3) \in \Omega^\delta}$$

Such an extension $\boldsymbol{\Theta} : \Omega^\delta \to \mathbb{E}^3$ of $\boldsymbol{\theta} : \omega \to \mathbb{E}^3$ is then used in the *left-hand side* of the above nonlinear 3d-Korn inequality. This explains why the vector field $\boldsymbol{a}_3$, resp. $\tilde{\boldsymbol{a}}_3$, appears together with the immersion $\boldsymbol{\theta}$, resp. $\tilde{\boldsymbol{\theta}}$, in the *left-hand side* of the nonlinear Korn inequality on a surface (each mapping $\tilde{\boldsymbol{\theta}}$ is similarly extended to a mapping $\tilde{\boldsymbol{\Theta}}$).



$\mathbf{a}_3(\mathbf{y})$
Θ
θ
2δ
$S=\theta(\overline{\omega})$
$\Theta(\Omega^\delta) \subset \mathbb{E}^3$
x_3
ω
y
2δ
$\Omega^\delta = \omega \times]-\delta, \delta[\subset \mathbb{R}^3$

The *assumptions*

$$|\tilde{R}_1| \geq \varepsilon \text{ and } |\tilde{R}_2| \geq \varepsilon \text{ a.e. in } \omega,$$

imply that

$$|\tilde{H}| = \left|\frac{1}{2}\left(\frac{1}{\tilde{R}_1} + \frac{1}{\tilde{R}_2}\right)\right| \leq \frac{1}{\varepsilon} \text{ and } |\tilde{K}| = \left|\frac{1}{\tilde{R}_1 \tilde{R}_2}\right| \leq \frac{1}{\varepsilon^2}$$

Hence the relations

$$\det \boldsymbol{\nabla\Theta}(y, x_3) = \sqrt{a(y)}\sqrt{1 - 2H(y)x_3 + K(y)x_3^2} \text{ for all } (y, x_3) \in \overline{\omega} \times \mathbb{R},$$

$$\det \boldsymbol{\nabla\tilde{\Theta}}(y, x_3) = \sqrt{\tilde{a}(y)}\sqrt{1 - 2\tilde{H}(y)x_3 + \tilde{K}(y)x_3^2} \text{ for a.a. } (y, x_3) \in \omega \times \mathbb{R}$$

show that $\det \boldsymbol{\nabla\Theta} > 0$ *in* $\overline{\Omega^\delta} = \overline{\omega} \times [-\delta, \delta]$ *and* $\det \boldsymbol{\nabla\tilde{\Theta}} > 0$ *a.e. in* $\Omega^\delta = \omega \times \,]-\delta, \delta[$ *if* $\delta = \delta(\varepsilon) > 0$ *is small enough*, so that the "nonlinear 3d-Korn inequality" can be applied.

The metric tensor $(g_{ij})_{i,j=1}^3 \stackrel{\text{def}}{=} \boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta}$ of the extension $\boldsymbol{\Theta} : \Omega^\delta \to \mathbb{E}^3$ of $\boldsymbol{\theta} : \omega \to \mathbb{E}^3$ appearing in the *right-hand side* of the above nonlinear 3d-Korn inequality is then given by

$$\boxed{\begin{aligned} g_{\alpha\beta}(y, x_3) &= a_{\alpha\beta}(y) - 2x_3 b_{\alpha\beta}(y) + x_3^2 c_{\alpha\beta}(y), \\ g_{i3}(y, x_3) &= \delta_{i3} \ \text{ for all } (y, x_3) \in \Omega^\delta = \omega \times \left]-\delta, \delta\right[\end{aligned}}$$

This explains the appearance of the *third fundamental form* in the *right-hand side* of the Korn inequality on a surface, with a similar expression for the metric tensor $\boldsymbol{\nabla\tilde{\Theta}}^T\boldsymbol{\nabla\tilde{\Theta}}$ of the extension $\boldsymbol{\tilde{\Theta}}$ of $\boldsymbol{\tilde{\theta}}$.

Question: Can we dispose of the *third fundamental forms* in the above nonlinear Korn inequalities on a surface?

Answer: *Yes*, if we make appropriate *additional assumptions*, as follows: Let:

$$\boldsymbol{A} \stackrel{\text{def}}{=} (a_{\alpha\beta}),\ \boldsymbol{B} \stackrel{\text{def}}{=} (b_{\alpha\beta}),\ \boldsymbol{C} \stackrel{\text{def}}{=} (c_{\alpha\beta})$$
$$\tilde{\boldsymbol{A}} \stackrel{\text{def}}{=} (\tilde{a}_{\alpha\beta}),\ \tilde{\boldsymbol{B}} \stackrel{\text{def}}{=} (\tilde{b}_{\alpha\beta}),\ \tilde{\boldsymbol{C}} \stackrel{\text{def}}{=} (\tilde{c}_{\alpha\beta})$$

Then $\boldsymbol{C} = \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}$ and $\tilde{\boldsymbol{C}} = \tilde{\boldsymbol{B}}\tilde{\boldsymbol{A}}^{-1}\tilde{\boldsymbol{B}}$, so that

$$\tilde{\boldsymbol{C}} - \boldsymbol{C} = \tilde{\boldsymbol{B}}\tilde{\boldsymbol{A}}^{-1}\Big((\tilde{\boldsymbol{B}} - \boldsymbol{B}) - (\tilde{\boldsymbol{A}} - \boldsymbol{A})\boldsymbol{A}^{-1}\boldsymbol{B}\Big) + (\tilde{\boldsymbol{B}} - \boldsymbol{B})\boldsymbol{A}^{-1}\boldsymbol{B}.$$

Consequently,

$$\|(\tilde{c}_{\alpha\beta} - c_{\alpha\beta})\|_{L^p(\omega;\mathbb{S}^2)} = \|\tilde{\boldsymbol{C}} - \boldsymbol{C}\|_{L^p(\omega;\mathbb{S}^2)}$$
$$\leq \|\tilde{\boldsymbol{B}}\tilde{\boldsymbol{A}}^{-1}\|_{L^\infty(\omega;\mathbb{M}^2)}\Big\{\|(\tilde{b}_{\alpha\beta} - b_{\alpha\beta})\|_{L^p(\omega;\mathbb{S}^2)} + c(\boldsymbol{\theta})\|(\tilde{a}_{\alpha\beta} - a_{\alpha\beta})\|_{L^p(\omega;\mathbb{S}^2)}\Big\}$$
$$+ c(\boldsymbol{\theta})\|(\tilde{b}_{\alpha\beta} - b_{\alpha\beta})\|_{L^p(\omega;\mathbb{S}^2)}$$

So we can dispose of $\|(\tilde{c}_{\alpha\beta} - c_{\alpha\beta})\|_{L^p(\omega;\mathbb{S}^2)}$ if there exists a constant M such that

$$\|\tilde{\boldsymbol{B}}\tilde{\boldsymbol{A}}^{-1}\|_{L^\infty(\omega;\mathbb{M}^2)} \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{y\in\omega} |(\tilde{\boldsymbol{B}}\tilde{\boldsymbol{A}}^{-1})(y)| \leq M$$

for all the "admissible" immersions $\tilde{\boldsymbol{\theta}}$. Noting that the matrix norm $|\tilde{\boldsymbol{B}}\tilde{\boldsymbol{A}}^{-1}(y)|$ is related to the norm of the *Weingarten map of the tangent plane* $\mathbb{T}_y\tilde{\boldsymbol{\theta}}$ to the surface $\tilde{\boldsymbol{\theta}}(\omega)$ at the point $\tilde{\boldsymbol{\theta}}(y)$ when the vectors of $\mathbb{T}_y\tilde{\boldsymbol{\theta}}$ are expanded over its covariant basis, one finds that, for almost all $y \in \omega$,

$$\boxed{|(\tilde{\boldsymbol{B}}\tilde{\boldsymbol{A}}^{-1})(y)| \leq |\tilde{\boldsymbol{A}}(y)|^{1/2}\, |\tilde{\boldsymbol{A}}(y)^{-1}|^{1/2} \max\left\{\frac{1}{|\tilde{R}_1(y)|}, \frac{1}{|\tilde{R}_2(y)|}\right\}}$$

These considerations yield the following "improved" nonlinear Korn inequalities on a surface.

P.G. Ciarlet, M. Malin & C. Mardare, in preparation.

THEOREM: $\omega \subset \mathbb{R}^2$ *bounded, open, connected; Lipschitz-continuous boundary. Let* $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega}; \mathbb{E}^3)$*: immersion such that* $\boldsymbol{a}_3 \in \mathcal{C}^1(\overline{\omega}; \mathbb{E}^3)$. *Assume that* $p > 1$.

Then, given any $\varepsilon > 0$, *there exists a constant* $c(p, \boldsymbol{\theta}; \varepsilon)$ *with the following property:*

Given any mapping $\widetilde{\boldsymbol{\theta}} \in W^{1,2p}(\omega; \mathbb{E}^3)$ *such that* $\tilde{\boldsymbol{a}}_1 \wedge \tilde{\boldsymbol{a}}_2 \neq 0$ *a.e. in* ω, $\tilde{\boldsymbol{a}}_3 \in W^{1,2p}(\omega; \mathbb{E}^3)$, *and*

$$\boxed{\begin{gathered} |\tilde{R}_1| \geq \varepsilon \text{ and } |\tilde{R}_2| \geq \varepsilon \text{ a.e. in } \omega, \\ |(\tilde{a}_{\alpha\beta})| \leq \frac{1}{\varepsilon} \text{ and } |(\tilde{a}_{\alpha\beta})^{-1}| \leq \frac{1}{\varepsilon} \text{ a.e. in } \omega, \end{gathered}}$$

there exists a mapping $\tilde{\boldsymbol{\theta}}^\sharp$ *isometrically equivalent to* $\tilde{\boldsymbol{\theta}}$ *such that the following* **nonlinear Korn inequality on a surface** *holds:*

$$\boxed{\begin{aligned} &\|\tilde{\boldsymbol{\theta}}^\sharp - \boldsymbol{\theta}\|_{W^{1,p}(\omega;\mathbb{E}^3)} + \|\tilde{\boldsymbol{a}}_3^\sharp - \boldsymbol{a}_3\|_{W^{1,p}(\omega;\mathbb{E}^3)} \\ &\leq c(p, \boldsymbol{\theta}, \varepsilon)\Big\{\|(\tilde{a}_{\alpha\beta} - a_{\alpha\beta})\|_{L^p(\omega;\mathbb{S}^2)} + \|(\tilde{b}_{\alpha\beta} - b_{\alpha\beta})\|_{L^p(\omega;\mathbb{S}^2)}\Big\} \end{aligned}} \quad \square$$

Comments. 1. The *assumptions*

$$|\tilde{R}_1| \geq \varepsilon \text{ and } |\tilde{R}_2| \geq \varepsilon \text{ a.e. in } \omega$$

mean that the deformed surfaces $\tilde{\boldsymbol{\theta}}(\omega)$ should "uniformly stay away from having folds" (corresponding to the vanishing of a principal radius of curvature).

2. The *assumptions*

$$|(\tilde{a}_{\alpha\beta})| \leq \frac{1}{\varepsilon} \text{ and } |(\tilde{a}_{\alpha\beta})^{-1}| \leq \frac{1}{\varepsilon} \text{ a.e. in } \omega$$

mean that the "infinitesimal length" along the deformed surfaces $\tilde{\boldsymbol{\theta}}(\omega)$ should uniformly satisfy

$$c_0 \left|\frac{d\boldsymbol{f}}{dt}(t)\right| dt \leq \sqrt{\tilde{a}_{\alpha\beta}(\boldsymbol{f}(t))\frac{df^\alpha}{dt}(t)\frac{df^\beta}{dt}(t)}\, dt \leq c_1 \left|\frac{d\boldsymbol{f}}{dt}(t)\right| dt$$

for some constants $c_0 > 0$ and $c_1 > 0$.

3. The mappings $\boldsymbol{\theta}$ and $\tilde{\boldsymbol{\theta}}^\sharp$ appearing in the above nonlinear Korn inequalities, or the mappings $\boldsymbol{\theta}$ and $\boldsymbol{\theta}^{\sharp k}$ appearing in the sequential continuity theorem, *do not have the same regularity*: Even though the estimates are with respect to the norm $\|\cdot\|_{W^{1,p}(\omega;\mathbb{E}^3)}$ of the differences $\tilde{\boldsymbol{\theta}}^\sharp - \boldsymbol{\theta}$ and $\tilde{\boldsymbol{a}}_3^\sharp - \boldsymbol{a}_3$, the immersions $\boldsymbol{\theta}$ and its unit normal vector field $\boldsymbol{a}_3$ are assumed to be in the space $\mathcal{C}^1(\overline{\omega};\mathbb{E}^3)$; this higher regularity assumption seems to be unavoidable.

4. In the same vein, *there is no existence theorem* guaranteeing the existence of an immersion $\boldsymbol{\theta} \in W^{1,2p}(\omega;\mathbb{E}^3)$ with $\boldsymbol{a}_3 \in W^{1,2p}(\omega;\mathbb{E}^3)$ under the assumption that the fundamental forms are in the space $L^p(\omega;\mathbb{S}^2)$ for some $p > 1$; this is why the mappings $\boldsymbol{\theta}$ and $\tilde{\boldsymbol{\theta}}$ are assumed to exist a priori.

In this respect, the recourse to *higher-order Sobolev spaces* seems to be unavoidable; cf. Section 9.

5. One can also establish an estimate of the difference between $\boldsymbol{\theta}$ and $\tilde{\boldsymbol{\theta}}$, i.e., without having to introduce a mapping $\tilde{\boldsymbol{\theta}}^\sharp$ isometrically equivalent to $\tilde{\boldsymbol{\theta}}$ if appropriate *terms with lower regularity* than in the left-hand side are *added to the right-hand side*. For instance, under the *same* assumptions as in the previous theorem, the following **nonlinear Korn inequality on a surface** holds:

$$
\boxed{
\begin{aligned}
&\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{W^{1,p}(\omega;\mathbb{E}^3)} + \|\tilde{\boldsymbol{a}}_3 - \boldsymbol{a}_3\|_{W^{1,p}(\omega;\mathbb{E}^3)} \\
&\qquad \leq c(p, \boldsymbol{\theta}, \varepsilon)\big\{\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{L^p(\omega;\mathbb{E}^3)} + \|\tilde{\boldsymbol{a}}_3 - \boldsymbol{a}_3\|_{L^p(\omega;\mathbb{E}^3)} \\
&\qquad\qquad + \|(\tilde{a}_{\alpha\beta} - a_{\alpha\beta})\|_{L^p(\omega;\mathbb{S}^2)} + \|(\tilde{b}_{\alpha\beta} - b_{\alpha\beta})\|_{L^p(\omega;\mathbb{S}^2)}\big\}
\end{aligned}
}
$$

Likewise, if, *in addition to* the assumptions of the previous theorem, the mappings $\tilde{\boldsymbol{\theta}}$ are subjected to *boundary conditions* of the form

$$\boxed{\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} \text{ and } \tilde{\boldsymbol{a}}_3 = \boldsymbol{a}_3 \text{ on } \gamma_0,}$$

where γ_0 is a non-empty relatively open subset of the boundary of ω, the following **nonlinear Korn inequality on a surface** holds:

$$\boxed{\begin{aligned}&\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{W^{1,p}(\omega;\mathbb{E}^3)} + \|\tilde{\boldsymbol{a}}_3 - \boldsymbol{a}_3\|_{W^{1,p}(\omega;\mathbb{E}^3)} \\ &\leq c(p, \boldsymbol{\theta}, \varepsilon, \gamma_0)\Big\{\|(\tilde{a}_{\alpha\beta} - a_{\alpha\beta})\|_{L^p(\omega;\mathbb{S}^2)} + \|(\tilde{b}_{\alpha\beta} - b_{\alpha\beta})\|_{L^p(\omega;\mathbb{S}^2)}\Big\}\end{aligned}}$$

6. By *formal linearization "in a neighborhood of $\boldsymbol{\theta}$"*, such nonlinear Korn inequalities reduce to known *linear Korn inequalities on a surface*; cf. Section 8.

8. LINEAR KORN INEQUALITY ON A SURFACE

$\alpha, \beta, \ldots \in \{1,2\}$ and $i, j, \ldots \in \{1,2,3\}$

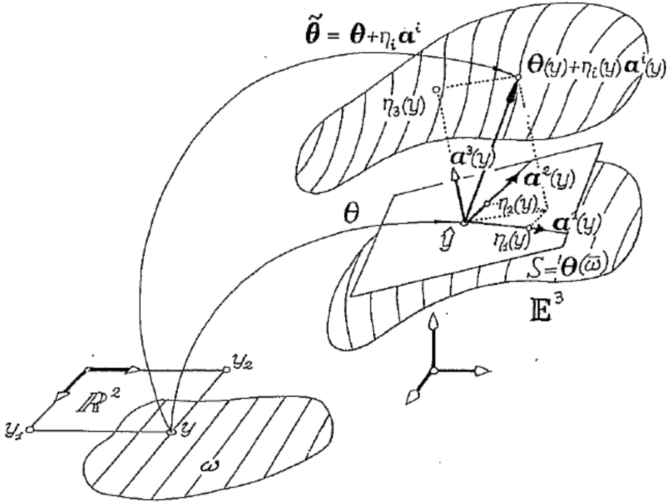
Contravariant basis: $\boldsymbol{a}^\alpha = a^{\alpha\beta}\boldsymbol{a}_\beta$, $(a^{\alpha\beta}) = (a_{\sigma\tau})^{-1}$, $\boldsymbol{a}^3 = \boldsymbol{a}_3$. Then
$\boldsymbol{a}^i \cdot \boldsymbol{a}_j = \delta^i_j$.
$\Gamma^\sigma_{\alpha\beta} = \boldsymbol{a}^\sigma \cdot \partial_\alpha \boldsymbol{a}_\beta$

Undeformed surface $\boldsymbol{\theta}(\omega)$: $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$
Deformed surface $\tilde{\boldsymbol{\theta}}(\omega)$: $(\tilde{a}_{\alpha\beta})$ and $(\tilde{b}_{\alpha\beta})$

$\widetilde{\boldsymbol{\eta}} = \eta_i \boldsymbol{a}^i : \omega \to \mathbb{E}^3$: displacement field
$\boldsymbol{\eta} = (\eta_i) : \omega \to \mathbb{R}^3$

Formal linearization means that the deformations $\tilde{\boldsymbol{\theta}}$ considered are of the form $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} + \tilde{\boldsymbol{\eta}}$ with $\tilde{\boldsymbol{\eta}}$, or equivalently $\boldsymbol{\eta}$, "small".

$[\ldots]^{\text{lin}} \stackrel{\text{def}}{=}$ linear part with respect to $\boldsymbol{\eta}$ in the expression ...



$\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} + \eta_i \boldsymbol{a}^i$
$\boldsymbol{\Theta}(y) + \eta_i(y)\boldsymbol{a}^i(y)$
$\eta_3(y)$
$\boldsymbol{a}^3(y)$
$\boldsymbol{a}^2(y)$
$\eta_2(y)$
θ
$\boldsymbol{a}^1(y)$
$\hat{y}$
$\eta_1(y)$
$S = \boldsymbol{\theta}(\overline{\omega})$
$\mathbb{E}^3$
y_2
$\mathbb{R}^2$
y_1
y
ω

Point of departure: there exists a constant c such that

$$\boxed{\begin{aligned}\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{W^{1,p}(\omega;\mathbb{E}^3)} + \|\tilde{\boldsymbol{a}}_3 - \boldsymbol{a}_3\|_{W^{1,p}(\omega;\mathbb{E}^3)} \\ \leq c\Big\{\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{L^p(\omega;\mathbb{E}^3)} + \|\tilde{\boldsymbol{a}}_3 - \boldsymbol{a}_3\|_{L^p(\omega;\mathbb{E}^3)} \\ + \|(\tilde{a}_{\alpha\beta} - a_{\alpha\beta})\|_{L^p(\omega;\mathbb{S}^2)} + \|(\tilde{b}_{\alpha\beta} - b_{\alpha\beta})\|_{L^p(\omega;\mathbb{S}^2)}\Big\}\end{aligned}}$$

Formal linearization:

$$\boxed{[\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}]^{\text{lin}} = \boldsymbol{\eta} \text{ and } [\tilde{\boldsymbol{a}}_3 - \boldsymbol{a}_3]^{\text{lin}} = -(\partial_\alpha \eta_3 + b_\alpha^\sigma \eta_\sigma)\boldsymbol{a}^\alpha}$$

$$\begin{aligned}
\gamma_{\alpha\beta}(\boldsymbol{\eta}) &\stackrel{\text{def}}{=} \frac{1}{2}\,[\tilde{a}_{\alpha\beta} - a_{\alpha\beta}]^{\text{lin}} = \frac{1}{2}(\partial_\alpha \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta + \partial_\beta \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\alpha) \\
&= \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - \Gamma_{\alpha\beta}^\sigma \eta_\sigma - b_{\alpha\beta}\eta_3
\end{aligned}$$

Linearized change of metric tensor

$$\begin{aligned}
\rho_{\alpha\beta}(\boldsymbol{\eta}) &\stackrel{\text{def}}{=} [\tilde{b}_{\alpha\beta} - b_{\alpha\beta}]^{\text{lin}} = (\partial_{\alpha\beta} \widetilde{\boldsymbol{\eta}} - \Gamma_{\alpha\beta}^\sigma \partial_\sigma \widetilde{\boldsymbol{\eta}}) \cdot \boldsymbol{a}_3 \\
&= \partial_{\alpha\beta}\eta_3 + A_{\alpha\beta}^{\sigma i}\partial_\sigma \eta_i + B_{\alpha\beta}^i \eta_i
\end{aligned}$$

Linearized change of curvature tensor

It is then easy to establish the *equivalence* (denoted $\simeq$) *of the following norms*, defined for sufficiently smooth vector fields $\boldsymbol{\eta} = (\eta_i) : \omega \to \mathbb{R}^3$:

$$\|[\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}]^{\mathrm{lin}}\|_{W^{1,p}(\omega;\mathbb{E}^3)} + \|[\tilde{\boldsymbol{a}}_3 - \boldsymbol{a}_3]^{\mathrm{lin}}\|_{W^{1,p}(\omega;\mathbb{E}^3)} \simeq \sum_\alpha \|\eta_\alpha\|_{W^{1,p}(\omega)} + \|\eta_3\|_{W^{2,p}(\omega)}$$

$$\|[\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}]^{\mathrm{lin}}\|_{L^p(\omega;\mathbb{E}^3)} + \|[\tilde{\boldsymbol{a}}_3 - \boldsymbol{a}_3]^{\mathrm{lin}}\|_{L^p(\omega;\mathbb{E}^3)} \simeq \sum_\alpha \|\eta_\alpha\|_{L^p(\omega)} + \|\eta_3\|_{W^{1,p}(\omega)}$$

Besides, by definition of the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$,

$$\|[(\tilde{a}_{\alpha\beta} - a_{\alpha\beta})]^{\mathrm{lin}}\|_{L^p(\omega;\mathbb{S}^2)} + \|[(\tilde{b}_{\alpha\beta} - b_{\alpha\beta})]^{\mathrm{lin}}\|_{L^p(\omega;\mathbb{S}^2)} \simeq \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{L^p(\omega)} + \sum_{\alpha,\beta} \|\rho_{\alpha\beta}(\boldsymbol{\eta})\|_{L^p(\omega)}$$

When $p = 2$, the resulting linear Korn inequality on a surface can be established directly; its proof hinges on *J.L. Lions lemma*:

THEOREM: LINEAR KORN INEQUALITY ON A SURFACE:
$\omega \subset \mathbb{R}^2$ *bounded, open, connected; Lipschitz-continuous boundary. Let* $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ *be an immersion. Then there exists a constant* c *such that*

$$\sum_\alpha \|\eta_\alpha\|_{H^1(\omega)} + \|\eta_3\|_{H^2(\omega)} \le c \Big\{ \sum_\alpha \|\eta_\alpha\|_{L^2(\omega)} + \|\eta_3\|_{H^1(\omega)} + \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{L^2(\omega)} + \sum_{\alpha,\beta} \|\rho_{\alpha\beta}(\boldsymbol{\eta})\|_{L^2(\omega)} \Big\}$$

for all $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$

(note that $\boldsymbol{\eta} \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ implies that $\gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ and $\rho_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$)

M. Bernadou & P.G. Ciarlet (1976)
M. Bernadou, P.G. Ciarlet & B. Miara: *J. Elasticity* (1994)
A. Blouza & H. Le Dret: *Quart. Appl. Math.* (1999)
P.G. Ciarlet & S. Mardare: *Math. Models Methods Appl. Sci.* (2001)

9. RECOVERY OF A SURFACE FROM ITS FUNDAMENTAL FORMS IN THE SOBOLEV SPACES $W^{2,p}(\omega), p > 2$

FUNDAMENTAL THEOREM OF SURFACE THEORY REVISITED, IN THE SPACE $W^{2,p}(\omega), p > 2$:
$\omega \subset \mathbb{R}^2$: *open, bounded, simply-connected; Lipschitz-continuous boundary. Let* $p > 2$ (so that $W^{1,p}(\omega) \hookrightarrow \mathcal{C}^0(\overline{\omega})$ and $W^{1,p}(\omega)$ is an algebra). *Let there be given* $(a_{\alpha\beta}) \in W^{1,p}(\omega, \mathbb{S}^2_>)$ *and* $(b_{\alpha\beta}) \in L^p(\omega, \mathbb{S}^2)$ *that satisfy the* **Gauß** *and* **Codazzi-Mainardi** *equations in* $\mathcal{D}'(\omega)$, *i.e., for all* $\varphi \in \mathcal{D}(\omega)$,

$$\boxed{\begin{aligned}&\int_\omega (\Gamma_{\alpha\beta\tau}\partial_\sigma\varphi - \Gamma_{\alpha\sigma\tau}\partial_\beta\varphi + \Gamma^\mu_{\alpha\beta}\Gamma_{\sigma\tau\mu}\varphi - \Gamma^\mu_{\alpha\sigma}\Gamma_{\beta\tau\mu}\varphi)\,\mathrm{d}y \\ &\qquad\qquad = \int_\omega (b_{\alpha\sigma}b_{\beta\tau} - b_{\alpha\beta}b_{\sigma\tau})\varphi\,\mathrm{d}y, \\ &\int_\omega (b_{\alpha\beta}\partial_\sigma\varphi - b_{\alpha\sigma}b_{\beta}\varphi + \Gamma^\mu_{\alpha\sigma}b_{\beta\mu}\varphi - \Gamma^\mu_{\alpha\beta}b_{\sigma\mu}\varphi)\,\mathrm{d}y = 0\end{aligned}}$$

$\square$

Then there exists $\boldsymbol{\theta} \in W^{2,p}(\omega; \mathbb{E}^3)$ *such that* $\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta} \neq \mathbf{0}$ *a.e. in* ω *and*

$$\boxed{\partial_\alpha\boldsymbol{\theta} \cdot \partial_\beta\boldsymbol{\theta} = a_{\alpha\beta} \text{ and } \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \frac{\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|} = b_{\alpha\beta} \text{ in } \omega}$$

Uniqueness holds up to isometric equivalence in $\mathbb{E}^3$*: All other solutions* $\boldsymbol{\theta}^\sharp$ *are:*

$$\boxed{\begin{aligned} &\boldsymbol{\theta}^\sharp(y) = \boldsymbol{a} + \boldsymbol{Q}\boldsymbol{\theta}(y), y \in \omega, \\ &\qquad\qquad \text{with } \boldsymbol{a} \in \mathbb{E}^3 \text{ and } \boldsymbol{Q} \in \mathbb{O}^3_+ \iff (\boldsymbol{\theta}^\sharp, \boldsymbol{\theta}) \in \mathcal{R} \end{aligned}}$$

$\square$

Brief outline of the proof: Same ideas as in the "classical" case, where $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in \mathcal{C}^1(\omega; \mathbb{S}^2)$, except that now $\mathcal{C}^m$-spaces are replaced by $W^{m,p}$-spaces.

(a) First, one seeks a solution $\boldsymbol{F} \in W^{1,p}(\omega; \mathbb{M}^3)$ to a Cauchy problem for a *Pfaff system* of the form

$$\partial_\alpha \boldsymbol{F} = \boldsymbol{F}\boldsymbol{\Gamma}_\alpha \text{ in } \omega \text{ with } \boldsymbol{\Gamma}_\alpha \in L^p(\omega; \mathbb{M}^3).$$

(b) Second, one seeks a solution $\boldsymbol{\theta} \in W^{2,p}(\omega; \mathbb{E}^3)$ to a Cauchy problem for a *Poincaré system* of the form

$$\partial_\alpha \boldsymbol{\theta} = \boldsymbol{a}_\alpha \text{ in } \omega \text{ with } \boldsymbol{a}_\alpha \in W^{1,p}(\omega; \mathbb{E}^3).$$

Such solutions exist, thanks to deep results due to:
S. Mardare, *J. Math. Pures Appl.* (2005)
S. Mardare, *Adv. Diff. Eqs.* (2007) $\square$