Recent Algorithmic Developments for the Markov Decision/Game Process

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Frontiers in Operations Research/Operations Management

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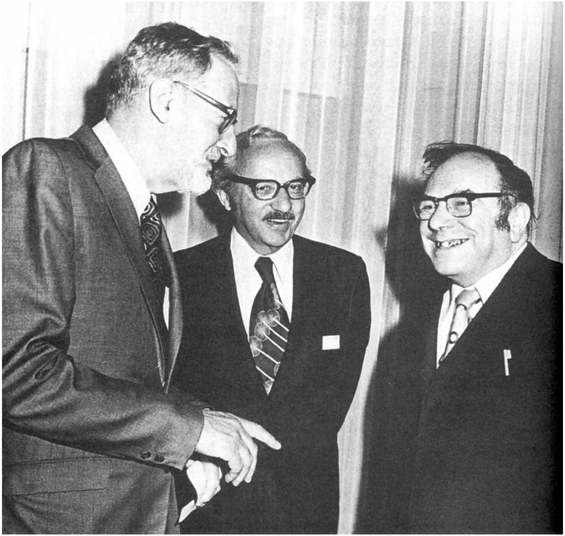
Linear Programming (LP)



Algebra of Linear Programming

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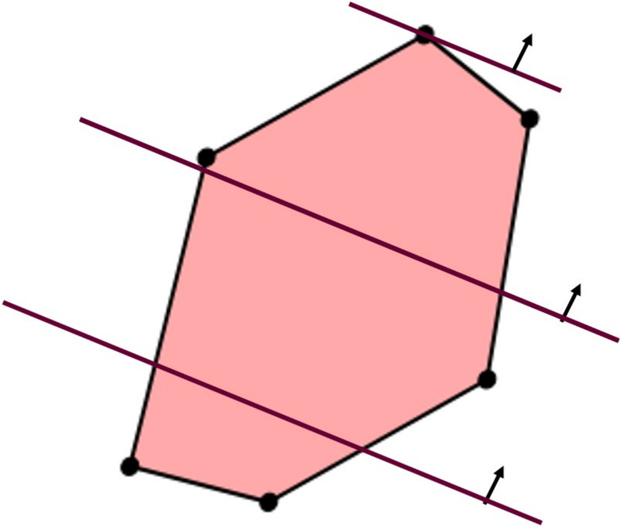
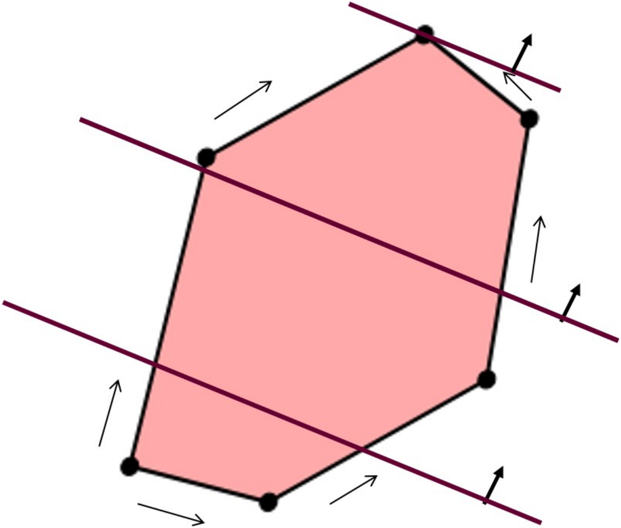
LP is a data-driven computation/decision model that is one of the most used computational problems.

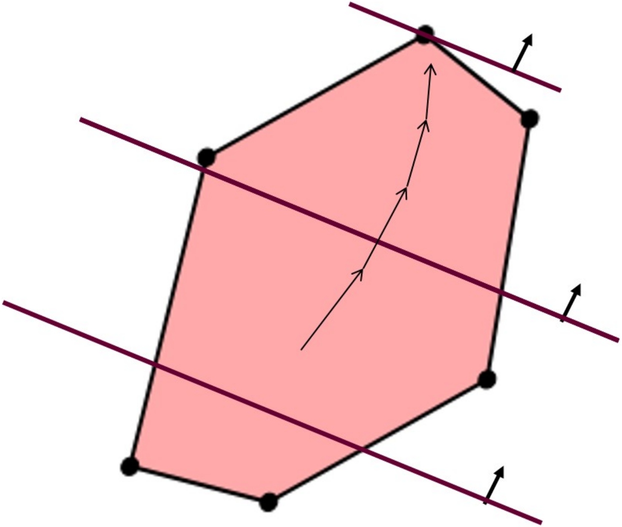
Geometry of Linear Programming



LP Algorithms: the Simplex Method



LP Algorithms: the Interior-Point Method



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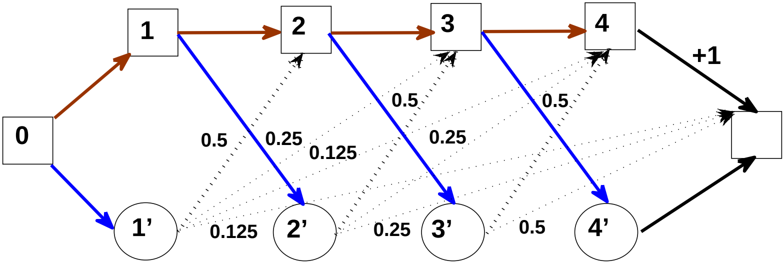
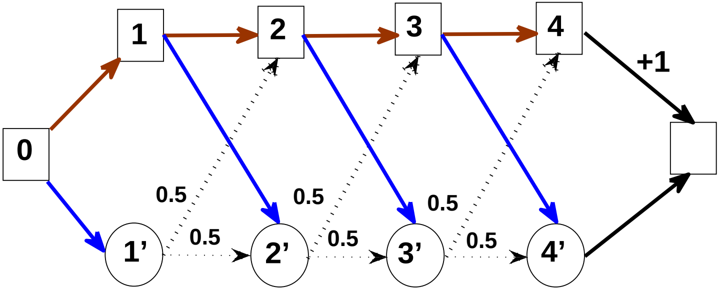
- MDGPs are useful for studying a wide range of optimization/game problems solved via dynamic programming, where it was known at least as early as the 1950s (cf. Shapley 1953, Bellman 1957).
- Modern applications include dynamic planning under uncertainty, reinforcement learning, social networking, and almost all other stochastic dynamic/sequential decision/game problems in Mathematical, Physical, Management and Social Sciences.

An MDGP Toy Example I



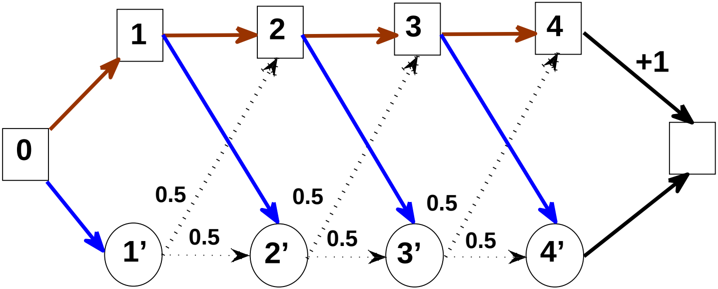
Actions are in red, blue and black; and all actions have zero cost except the state 4 to the termination state (Melekopoglou and Condon 1990). Which actions to take from every state to minimize the total cost?

Toy Example: Simplified Representation



Actions are in red, blue and black; and all actions have immediate zero cost except the state 4 to the termination state.

Toy Example: Game Setting



States $\{0, 1, 2\}$ minimize, while States $\{3, 4\}$ maximize.

The Markov Decision Process/Game continued

- At each time step, the process is in some state $i = 1, ..., m$, and the decision maker chooses an action $j \in \mathcal{A}_i$ that is available in state i, and giving the decision maker an immediate corresponding cost c_j.

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- But given state/action j, the distribution is conditionally independent of all previous states and actions; in other words, the state transitions of an MDP possess the Markov property.

MDP Stationary Policy and Cost-to-Go Value

- A stationary policy for the decision maker is a function $\pi = \{\pi_1, \pi_2, \cdots, \pi_m\}$ that specifies an action in each state, $\pi_i \in \mathcal{A}_i$, that the decision maker will always choose; which also lead to a cost-to-go value for each state

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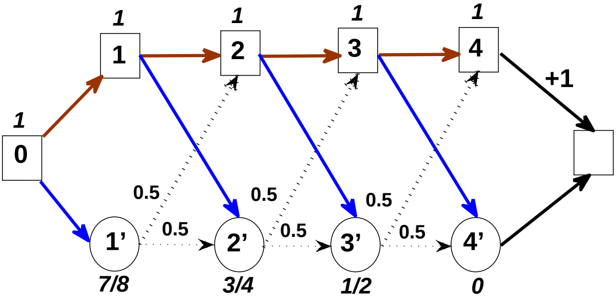
- If the states are partitioned into two sets, one is to minimize and the other is to maximize the discounted sum, then the process becomes a two-person turn-based zero-sum stochastic game.
- Typically, discount factor $\gamma = \frac{1}{1+\rho}$ where ρ is the interest rate, where we assume it is uniform among all actions.

The Cost-to-Go Values of the States



Cost-to-go values on each state when actions in red are taken (the current policy is not optimal).

The Optimal Cost-to-Go Value Vector

Let $\mathbf{y} \in \mathbf{R}^m$ represent the cost-to-go values of the m states, one entry for each state i, of a given policy.

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The MDP problem entails choosing the optimal value vector $\mathbf{y}^*$ such that it is the fixed point:

$$y_i^* = \min\{c_j + \gamma \mathbf{p}_j^T \mathbf{y}^*, \ \forall j \in \mathcal{A}_i\}, \ \forall i,$$

with optimal policy

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In the Game setting, the fixed point becomes:

$$y_i^* = \min\{c_j + \gamma \mathbf{p}_j^T \mathbf{y}^*, \ \forall j \in \mathcal{A}_i\}, \ \forall i \in I^-,$$

and

$$y_i^* = \max\{c_j + \gamma \mathbf{p}_j^T \mathbf{y}^*, \ \forall j \in \mathcal{A}_i\}, \ \forall i \in I^+.$$

The Linear Programming Form of the MDP

The fixed-point vector can be formulated as

$$
\begin{array}{llll}
\text{maximize}_{\mathbf{y}} & \sum_{i=1}^{m} y_i & & \\
\text{subject to} & y_1 & \leq & c_j + \gamma \mathbf{p}_j^T \mathbf{y}, \ \forall j \in \mathcal{A}_1 \\
& \ldots & \ldots & \ldots \\
& y_i & \leq & c_j + \gamma \mathbf{p}_j^T \mathbf{y}, \ \forall j \in \mathcal{A}_i \\
& \ldots & \ldots & \ldots \\
& y_m & \leq & c_j + \gamma \mathbf{p}_j^T \mathbf{y}, \ \forall j \in \mathcal{A}_m,
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where $\mathcal{A}_i$ represents all actions available in state i, and $\mathbf{p}_j$ is the state transition probabilities to all states when action j is taken.

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where $\mathcal{A}_i$ represents all actions available in state i, and $\mathbf{p}_j$ is the state transition probabilities to all states when action j is taken.

This is the Standard Dual LP form.

The Primal LP Form of the MDP

$$\begin{array}{lll} \text{minimize}_{\mathbf{x}} & \sum_{j=1}^{n} x_j & \\ \text{subject to} & \sum_{j=1}^{n}(e_{ij} - \gamma p_{ij})x_j & = 1, \ \forall i, \\ & x_j & \geq 0, \ \forall j. \end{array}$$

where $e_{ij} = 1$ when $j \in \mathcal{A}_i$ and 0 otherwise.

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When discount factor γ becomes γ_j, then the MDP has a non-uniform discount factors.

Algorithmic Events of the MDP Methods

- Shapley (1953) and Bellman (1957) developed a method called the Value-Iteration (VI) method to approximate the optimal state cost-to-go values and an approximate optimal policy.

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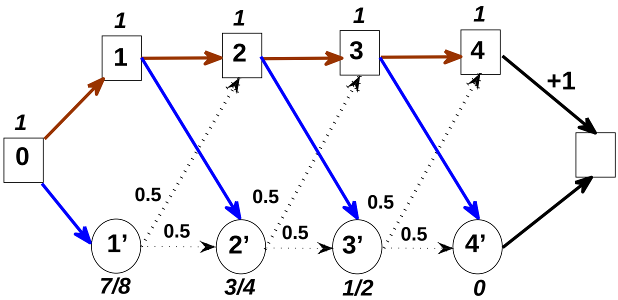
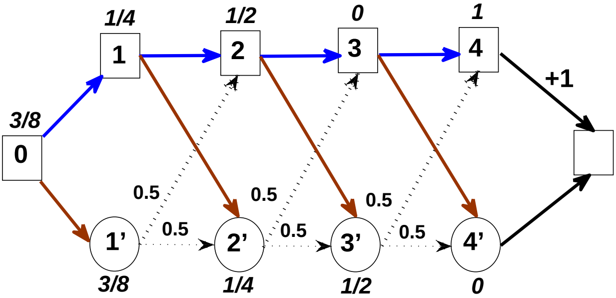
- de Ghellinck (1960), D'Epenoux (1960) and Manne (1960) showed that the MDP has an LP representation, so that it can be solved by the simplex method of Dantzig (1947) in finite number of steps, and the Ellipsoid method of Kachiyan (1979) in polynomial time.

The Policy Improvement



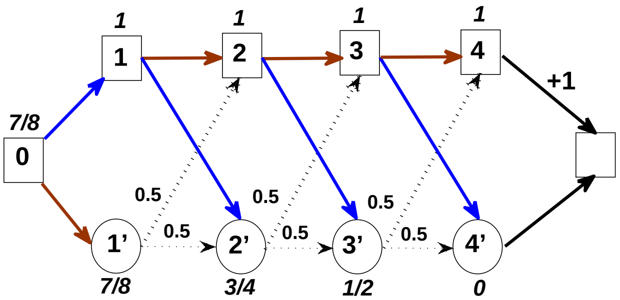
Cost-to-go values on each state when actions in red are taken (the current policy is not optimal), and each state (except State 4) would switch to blue actions.

The Policy Evaluation



New cost-to-go values on each state when the new set of actions are taken, which are colored in red.

The Simplex or Simple Policy Iteration: index rule



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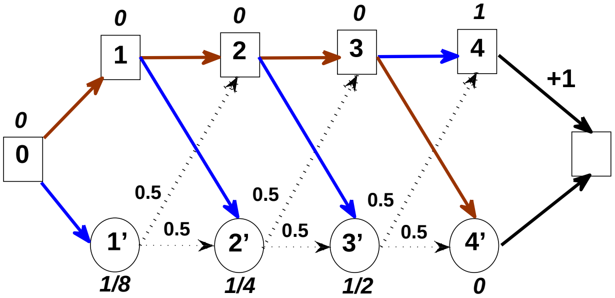
The Simplex or Simple Policy Iteration: greedy rule



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More Algorithmic Events of the MDP Methods

For the discounted MDP:

- Meister and Holzbaur in 1986 showed that the value iteration method generates an optimal policy in polynomial time when the discount γ is fixed, and Bertsekas (1987) and Tseng (1990) showed similar results.

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- However, the PI dominates the CIPM In practice.

Polynomial vs Strongly Polynomial

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- The proof of polynomial-time: when the gap between the objective value of the current policy (or BFS) and the optimal one is small than 2^{-L}, the current policy must be optimal.
- A proof of a strongly polynomial-time algorithm cannot rely on this gap argument – it has to be combinatorial.

Negative Results for the Simplex and Policy-Iteration Methods

- A negative result of Melekopoglou and Condon (1990) showed that a simple policy-iteration method, where in each iteration only the action for the state with the smallest index is updated, needs an exponential number of iterations to compute an optimal policy for a specific MDP problem regardless of discount rates.

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- Friedman (2011) developed an MDP example that the Zadeh pivot rule needs exponentially many steps.

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- In the past 50 years, many efforts have been made to resolve the worst-case complexity issue of the policy-iteration method or the Simplex method, and to answer the question: are they strongly polynomial-time algorithms?

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Complexity Theorem for MDP with Discount

- The classic simplex method (Dantzig pivoting rule) and the policy iteration method, starting from any policy, terminate in

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iterations (Y MOR10).

- The policy-iteration method actually terminates

$$\frac{n}{1-\gamma} \cdot \log\left(\frac{m}{1-\gamma}\right),$$

iterations with at most $O(m^2 n)$ operations per iteration (Hansen/Miltersen/Zwick ACM12).

High Level Ideas of the Proof

- Create a combinatorial event: a (non-optimal) action will never enter the (intermediate) policy again.

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- The event then repeats for another non-optimal state-action, and there are no more than $(n-m)$ non-optimal actions to eliminate.

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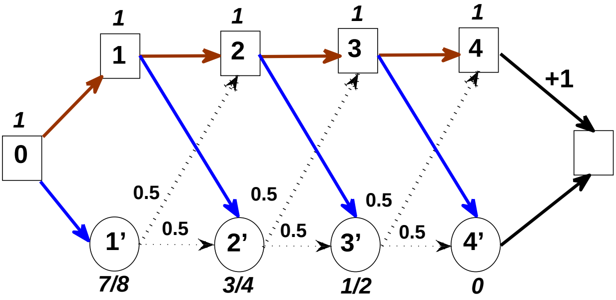
- Strategy-Iteration Method: One player continues policy iterations from the policy where the other player chooses the best-response action in every one of his or her state set.
- Hansen/Miltersen/Zwick ACM12 proved that the strategy iteration method also terminates

$$\frac{n}{1-\gamma} \cdot \log\left(\frac{m}{1-\gamma}\right)$$

iterations – the first strongly polynomial time algorithm when the discount factor is fixed.

Strategy-Iteration Method



Recall that states $\{0, 1, 2\}$ minimize, while States $\{3, 4\}$ maximize.

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- This can be exactly formulated as the a Turn-Based Two-Person Zero-Sum Game, so that the strategy iteration method also terminates

$$\frac{n}{1-\gamma} \cdot \log\left(\frac{n+m}{1-\gamma}\right)$$

iterations.

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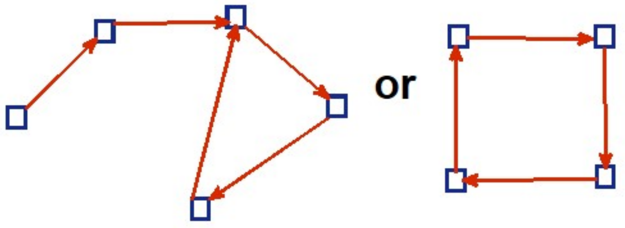
- Hansen/Miltersen/Zwick 15 were able to reduce a factor m from the bound.

High Level Ideas of the Proof I



- Each chosen action can be either a path-edge or cycle-edge of a policy, and the expected edge flow value is small when it is a path-edge and large when it is a cycle edge.

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- For the non-uniform discount case, there are n different edge flow value layers...

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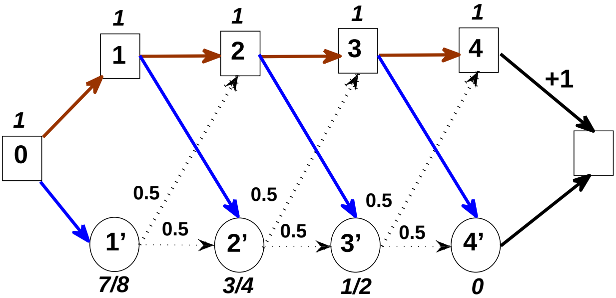
$$y_i^{k+1} = \max\{c_j + \gamma \mathbf{p}_j^T \mathbf{y}^k, \ \forall j \in \mathcal{A}_i\}, \ \forall i \in I^+.$$

The values inside the parenthesis are the so-called Q-values.

Value-Iteration Method of MDP



Assume discount $\gamma = 0.9$ and start the value vector $(1, 1, 1, 1, 1, 0)$. The next would be $(0.788, 0.675, 0.45, 0, 1, 0)$ for MDP, and $(0.788, 0.675, 0.45, 0.9, 1, 0)$ for MGP.

Sample Value-Iteration

- Rather than compute each quantity $\mathbf{p}_j^T \mathbf{y}^k$ exactly, we approximate it by sampling, that is, we construct a sparser sample distribution $\hat{\mathbf{p}}_j$ for the evaluation. (Thus, the method does not need to know $\mathbf{p}_j$ exactly).

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- We analyze this performance using Hoeffdings inequality and classic results on contraction properties of value iteration. Moreover, we improve the final result using Variance Reduction and Monotone Iteration.
- Variance Reduction enables us to update the Q-values so that the needed number of samples is decreased from iteration to iteration.

Sample Value-Iteration Results

Two results are developed (Sidford, Wang, Wu and Y [2017]):

- Knowing $\mathbf{p}_j$:

$$O\left((mn + \frac{n}{(1-\gamma)^3}) \log(\frac{1}{\epsilon}) \log(\frac{1}{\delta})\right)$$

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