L-functions: its past and future

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$S_3(n) = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4},$ *Aryabhata*$(476AD)$

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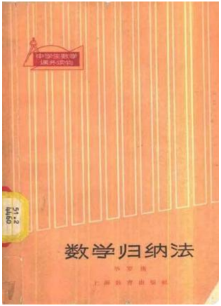

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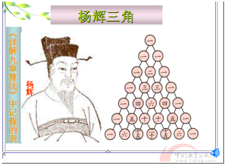
$$S_k(n) = \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j C_j^{k+1} B_j n^{k+1-j} = \frac{(B+n)^{k+1} - B^{k+1}}{k+1}.$$

Yang Hui – Pascal triangle



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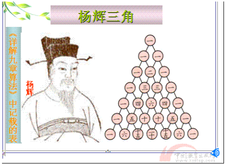

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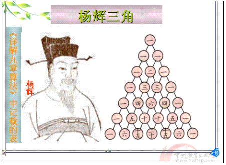

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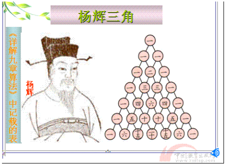

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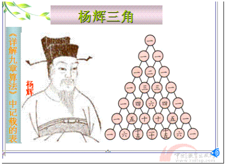

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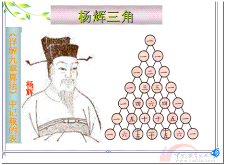

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Set $G(z, n) = \sum_{k=0}^{\infty} S_k(n) \frac{z^k}{k!}$.

proof

Set $G(z,n)=\sum_{k=0}^{\infty}S_k(n)\frac{z^k}{k!}$. Then

$$G(z,n)=\frac{1-e^{nz}}{e^{-z}-1}=\sum_{j=0}^{\infty}B_j\frac{(-z)^{j-1}}{j!}\sum_{i=1}^{\infty}-\frac{(nz)^i}{i!}$$

$$=\sum_{k=0}^{\infty}\frac{z^k}{k!}\frac{1}{k+1}\sum_{j=0}^{k}(-1)^jC_{k+1}^{j}B_jn^{k+1-j}.$$

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For any nice function $f(x)$, the sum $S(n) = \sum_{i=1}^{n} f(i)$ extends to a nice function $S(x)$.

The relation between $f(x)$ and $S(x)$ is given by difference equation:

$$f(x) = S(x) - S(x-1).$$

Euler formula

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$$f(x) = (1 - e^{-D})S(x) = \frac{1 - e^{-D}}{D} DS(x)$$

Thus we obtain a differential equation

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For $f(x) = x^k$ with $k \geq 0$ an integer, Euler's formula recovers Bernoulli's formula.

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For each real $k > 1$, how to evaluate the infinite sum:

$$\zeta(k) := S_{-k}(\infty) = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots ?$$

Basel problem

In 1735, Euler made the first major contribution to this problem: he solved the case $k = 2$ which was called Basel problem:

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$$\sin x = x \prod_{k=1}^{\infty} (1 - \frac{x^2}{k^2 \pi^2}).$$

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The argument also applies to symmetric functions of $1/\pi^2 k^2$ and thus gives a formula for $\zeta(2k)$. More precisely, two expansions of $\cot x = (\log \sin x)'$ will give

$$\zeta(2k) = \frac{(-1)^{k-1} B_{2k} (2\pi)^{2k}}{2(2k)!}$$

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This is a major refinement of the infiniteness of primes proved by Euclid (300 BC).

Euler zeta function for positive reals

Euler extended $\zeta(s)$ to reals in $(0, 1)$ by the following methods:

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He further extended it to negative of s when the limit exists:

$$\zeta(s) = \frac{1}{1-2^{1-s}} \lim_{x \to 1^-} (x - \frac{x^2}{2^s} + \frac{x^3}{3^s} - \cdots).$$

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In comparison with $\zeta(2k) = \frac{(-1)^{k-1} B_{2k} (2\pi)^{2k}}{2(2k)!}$, there is a "functional equation" between $\zeta(s)$ and $\zeta(1-s)$ for integers.

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For two integers N, a coprime to each other, there are infinitely many primes p such that $p \equiv a \mod N$.

Moreover,

$$\sum_{p \equiv a \mod N} \frac{1}{p} = \infty.$$

Fourier analysis on finite group

Let G be a finite abelian group with uniform probabilistic measure. Then there is an orthogonal decomposition

$$L^2(G) = \sum_{\chi \in \widehat{G}} \mathbb{C}\chi, \qquad \widehat{G} := \mathrm{Hom}(G, \mathbb{C}^\times).$$

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Thus for any function f on G, there is a Fourier expansion

$$f = \sum_{\chi} \langle f, \chi \rangle \chi, \qquad \langle f, \chi \rangle = \frac{1}{\# G} \sum_{g \in G} f(g) \bar{\chi}(g).$$

Apply to $G = (\mathbb{Z}/N\mathbb{Z})^\times$ and $f = \delta_a$, we obtain

$$\delta_a = \sum_\chi \langle \delta_a, \chi \rangle \chi.$$

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Consequently,

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When $\chi = 1$, $\sum_p \frac{\chi(p)}{p} = \infty$ by Euler, it suffices to show when $\chi \neq 1$

$$\sum_p \frac{\chi(p)}{p} \neq \infty$$

Dirichlet L-function

For χ a character of $(\mathbb{Z}/N\mathbb{Z})^\times$, Dirichlet introduce his function

$$L(\chi, s) = \sum_n \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1 - \frac{\chi(p)}{p^s}}, \qquad s > 1.$$

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$$\log L(\chi, 1) = \sum \frac{\chi(p)}{p} + O(1).$$

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Then

$$\log L(\chi, 1) = \sum \frac{\chi(p)}{p} + O(1).$$

So is suffices to show that $L(\chi, 1)$ is finite and nonzero.

Fourier analysis on additive group

Now we apply Fourier analysis to $M = \mathbb{Z}/N\mathbb{Z}$.

$$L^2(M) = \sum_{\widehat{M}} \mathbb{C}\psi, \qquad \widehat{M} := \mathrm{Hom}(M, \mathbb{C}^\times).$$

Extend $G \subset M$, then we have Fourier expansion

$$\chi = \sum \langle \chi, \psi \rangle \psi, \qquad \langle \chi, \psi \rangle = \frac{1}{N} \sum_{m \in M} \psi(m) \bar{\chi}(m).$$

This implies

$$L(\chi, s) = \sum_{\psi} \langle \chi, \psi \rangle L(\psi, s), \quad L(\psi, s) = \sum_{n} \frac{\psi(n)}{n^s}$$

Dirichlet special value formula

The value $L(\psi, 1)$ is computable:

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Example: $N = 4$, $\chi : (\mathbb{Z}/4\mathbb{Z})^\times \to \{\pm 1\}$, then

$$L(\chi, 1) = 1 - \frac{1}{3} + \frac{1}{5} \cdots = \frac{\pi}{4}.$$

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Riemann discovered several extremely important properties using **Fourier analysis** on $\mathbb{R}$ and $\mathbb{R}^\times$.

Continuation and functional equation

Using $\Gamma(s) := \langle e^{-x}, x^{s-1}\rangle = \int_0^\infty e^{-x}x^s \frac{dx}{x}$, one obtains $\zeta(s)$ as the Mellin transform of a theta function:

$$\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty f(t)t^{s/2}\frac{dt}{t}, \qquad f(t) := \sum_{n=1}^\infty e^{-\pi n^2 t}.$$

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Writing $\int_0^\infty = \int_0^1 + \int_1^\infty$ and using Poisson summation formula, we have

$$\xi(s) = \int_1^\infty f(t)(t^{s/2} + t^{(1-s)/2}) \frac{dt}{t} - \left(\frac{1}{s} + \frac{1}{1-s} \right).$$

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This gives the meromorphic continuation and the functional equation $\xi(s) = \xi(1-s)$. This somehow explains the relation between $\zeta(1-k)$ and $\zeta(k)$ given by Euler.

There is another product formula of $\zeta(s)$ in terms of its zeros (as $\sin x$).

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Riemann Hypothesis is equivalent to an asymptotic formula for the number of primes

$$\pi(x) := \#\{p \leq x\} = \int_2^x \frac{dt}{\log t} + O_\epsilon(x^{\frac{1}{2}+\epsilon})$$

which is a conjectural stronger form of the prime number theorem.

A general Dirichlet series takes a form

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \frac{1}{F_p(s)} \qquad a_n \in \mathbb{C}.$$

with F_p a polynomial of p^{-s} with leading coefficient 1 and of a fixed degree.

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It should have meromorphic continuation to the complex plane, and satisfies a functional equation.
How to find a general L-series?

Dedekind

In 1877, Dedekind generalized Dirichlet's work to number fields K.

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{\mathrm{N}(\mathfrak{a})^s} = \prod_{\mathfrak{p}} \frac{1}{1 - \mathrm{N}\mathfrak{p}^{-s}}.$$

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He has managed to prove the meromorphic continuation and a functional equation.
In particular, $\zeta_K(s)$ has a simple pole at $s = 1$ with residue given by

$$\frac{2^{r_1}(2\pi)^{r_2} hR}{w\sqrt{d_K}}.$$

Hecke and Tate on Grossen character

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Analytic properties of $L(s)$ can be studied using Fourier analysis on GL_n and $M_{n\times n}$.

Relation with arithmetic geometry

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A web of conjectures assert that the special values of these L-functions often give crucial information about the Diophantine properties of the varieties, such as BSD, Tate, etc.
Riemann hypothesis, Langlands program, and special values of L-series are three major topics of number theory in the 21st century.