EQUILIBRIUM AND FORCES

FROM ARISTOTLE TO LAGRANGE

jean.salencon@m4x.org

MECHANICS ?

FROM ARISTOTLE (384-322 BC)

Mechanics?

"Miraculously some facts occur in physics whose causes are unknown; that is, those artifices that appear to transgress Nature in favour of man…Thus, when it is necessary to do something that goes beyond Nature, the difficulties can be overcome with the assistance of art. ***Mechanics is the name of the art that*** ***helps us over these difficulties; as the poet Antiphon put it,*** ***"Art brings the victory that Nature impedes."***

Quaestionae Mechnicae, Mechanical Problems

(Suspected to be apocryphal)

Mechanics

Borrowing from Middle French *machine*, from Latin *māchina* (“a machine, engine, contrivance, device, stratagem, trick”), **from Ancient Greek *μᾱχᾰνᾱ́* (mākhanā́), Doric form of *μηχᾰνή* (mēkhanḗ, “a machine, engine, contrivance, device”).**

Wiktionary.org

Mechanics

Mechanics *noun*

1 the scientific study of motion and force.

2 the science of machinery

The Oxford Paperback Dictionary, p. 500.

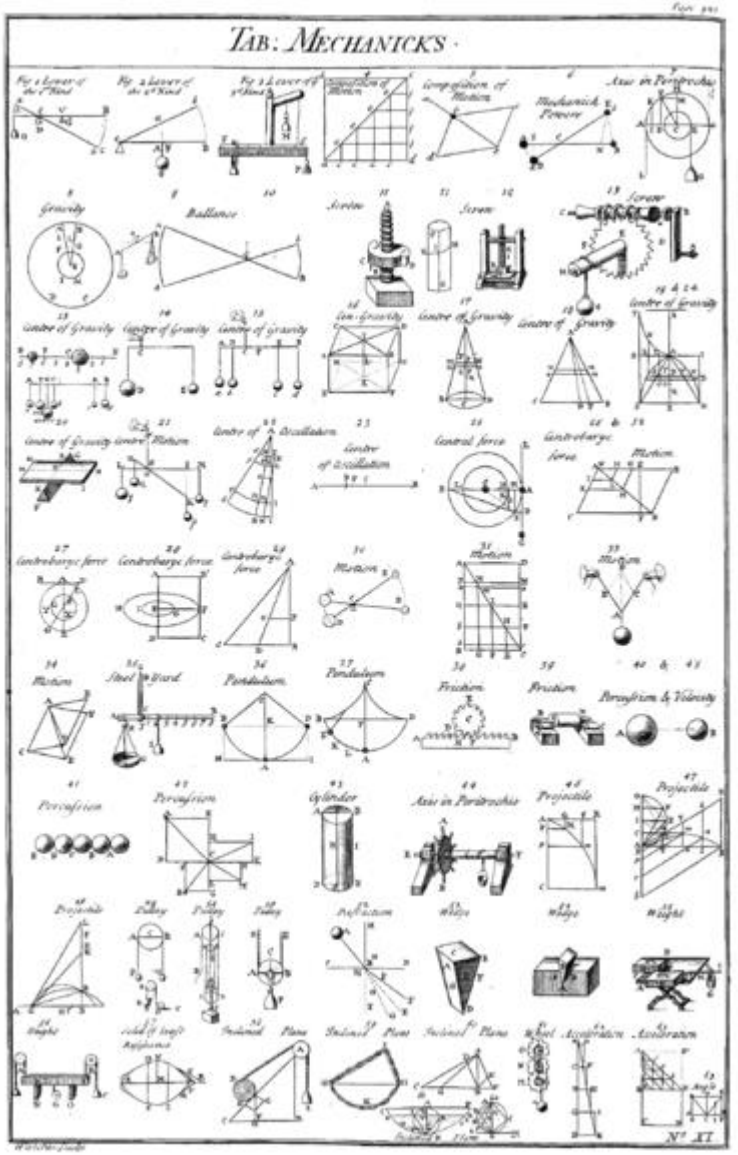
EQUILIBRIUM vs MOTION

The Simple Machines

“Early theoretical thinking about statics and mechanics took as its references particular objects, things like the lever, used since ancient times as necessary tools.”

Benvenuto (1940-1998)

The Simple Machines



Chambers, Ephraim (1728), "Table of Mechanicks", Cyclopædia, A Useful Dictionary of Arts and Sciences, London, England, Volume 2, p. 528, Plate 11.

ARISTOTLE's Marvellous "original cause"

"…Among the problems included in this class are included those concerned with the lever. For it is strange that a great weight can be moved by a small force, …

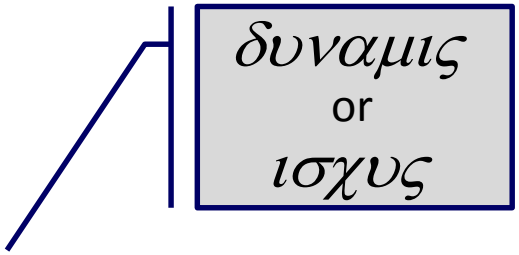
Now the original cause of all such phenomena is the circle; and this is natural, for it is in no way strange that something remarkable should result from something more remarkable, and the most remarkable fact is the combination of opposites with each other."

"Therefore, as has been said before, there is nothing strange in the circle being the first of all marvels."

Quaestionae Mechnicae, Mechanical Problems

(Suspected to be apocryphal)

ARISTOTLE's Rules of proportion



"Power Law"

"Then, A the movement have moved B a distance G in a time D, then in the same time the same force A will move ½ B twice the distance G, and in ½ D it will move ½ B the whole distance for G: thus the rules of proportion will be observed."

Weight Distance

$$A = \frac{B \times G}{D}$$

Power Time

A MAJOR AMBIGUITY

to be definitely ruled out by DESCARTES

http://classics.mit.edu/Aristotle/physics.7.vii.html

ARCHIMEDES (287-212 BC)

de Planorum Æquilibriis, On the Equilibrium of Planes

Follows Euclid's footsteps

Sets up a few axioms:
simple abstractions derived from
everybody and everyday experience

Then proceeds step-by-step
deriving new results

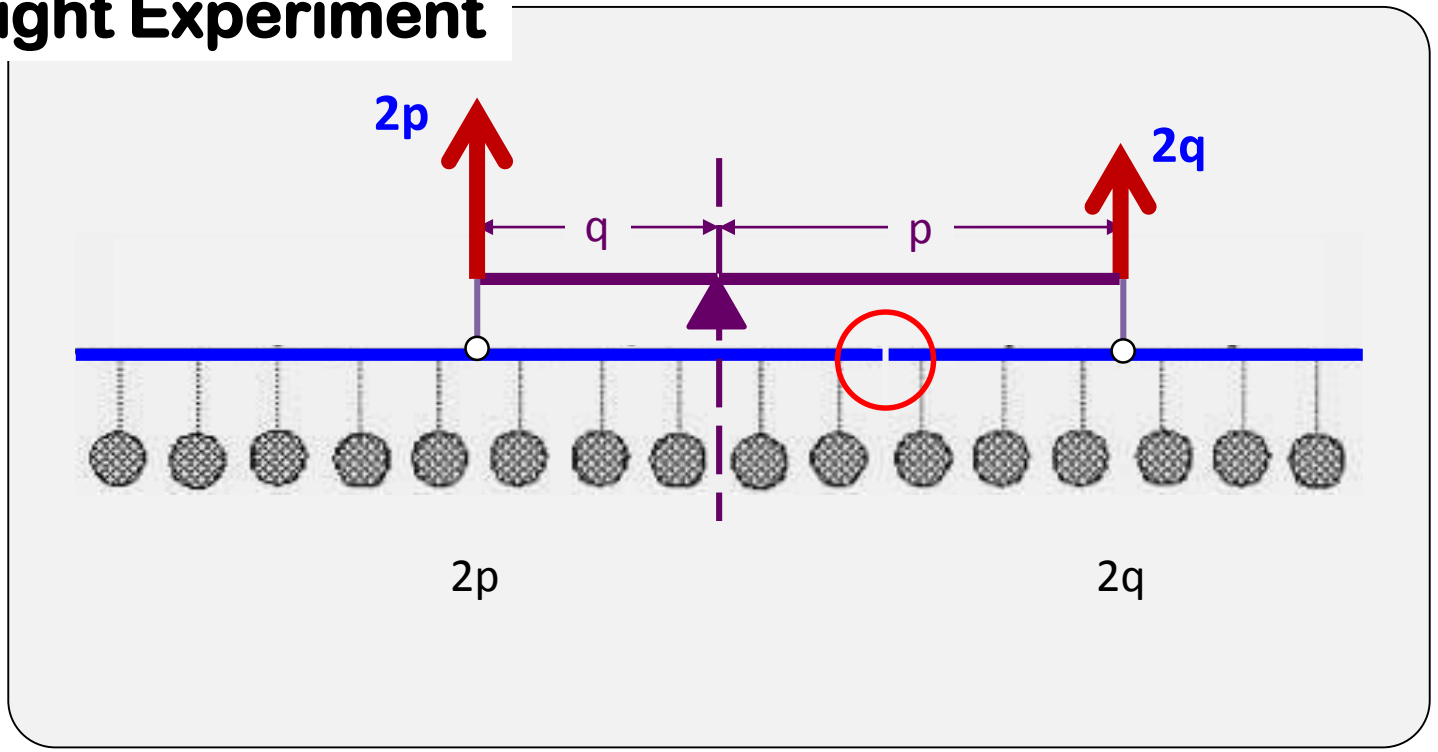
No reference to any governing Rule or Principle

ARCHIMEDES

Proof of the Principle of the lever

Equilibrium of the lever with arms of equal length

Statical Thought Experiment

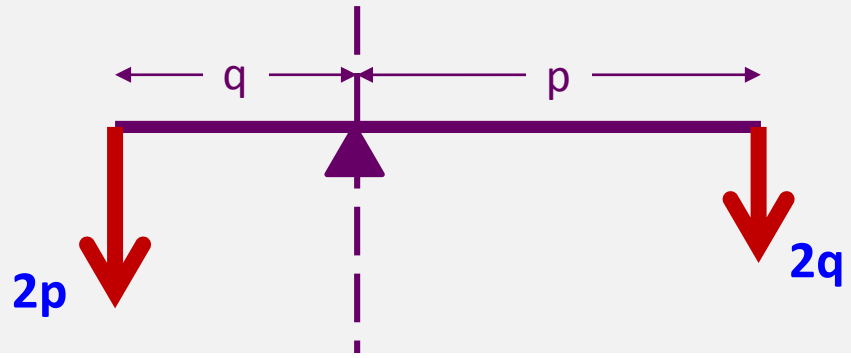


ARCHIMEDES

Proof of the Principle of the lever

Equilibrium of the lever with arms of equal length

No reference to motion



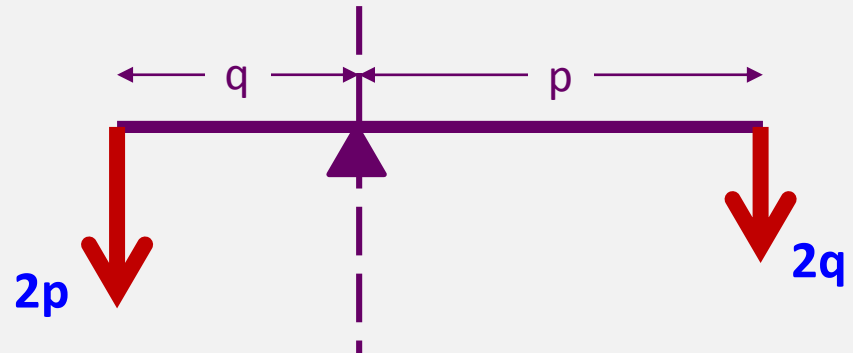
The weight 2p with lever arm q
is equilibrated
by the weight 2q with lever arm p

ARCHIMEDES

Proof of the Principle of the lever

Equilibrium of the lever with arms of equal length

No overruling general principle



The weight 2p with lever arm q
is equilibrated
by the weight 2q with lever arm p

DUHEM's comments

P. Duhem (1861-1916)

ARCHIMEDES

Plainly explains *"Quod ita sit"* **WHAT**

But **not** *"Cur ita sit"* **WHY**

ARISTOTLE

"This insight is, indeed, the seed from which will come out, through a twenty century development, the powerful ramifications of the Principle of virtual velocities"

ARCHIMEDES

Plainly explains *"Quod ita sit"* **WHAT**

But **not** *"Cur ita sit"* **WHY**

Two fundamental pathways in the history of mechanics

ARISTOTLE

"This insight is, indeed, the seed from which will come out, through a twenty century development, the powerful ramifications of the Principle of virtual velocities"

STEVIN (1548-1620)

sometimes called Stevinus, Flemish/Dutch/Netherlandish mathematician, physicist and engineer.

https://en.wikipedia.org/wiki/Simon_Stevin

STEVIN

Discards Aristotle's marvel argument based upon movement and circular trajectories

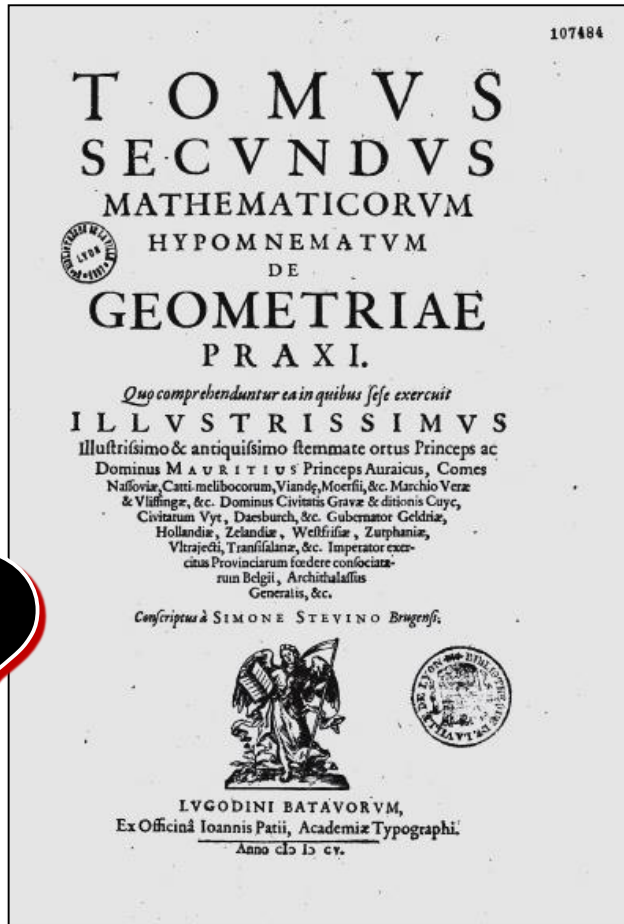
"Weights that are in equilibrium are motionless"

"Weights that are in equilibrium do not move along circles"

A conceptual difficulty to be overcome

Why should the analysis of equilibrium refer to motion

?


107484

TOMVS
SECVNDVS
MATHEMATICORVM
HYPOMNEMATVM
DE
GEOMETRIAE
PRAXI.

Quo comprehenduntur ea in quibus ſeſe exercuit

ILLVSTRISSIMVS
Illuſtriſsimo & antiquiſsimo ſtemmate ortus Princeps ac Dominus MAURITIUS Princeps Auraicus, Comes Naſſoviæ, Catti-melibocorum, Viandę, Moerſii, &c. Marchio Veræ & Vliſſingæ, &c. Dominus Civitatis Gravæ & ditionis Cuyc, Civitatum Vyt, Daesburch, &c. Gubernator Geldriæ, Hollandiæ, Zelandiæ, Weſtfriſiæ, Zurphaniæ, Vltrajecti, Tranſiſalanæ, &c. Imperator exercitus Provinciarum fœdere confociatarum Belgii, Archithalaſſis Generalis, &c.

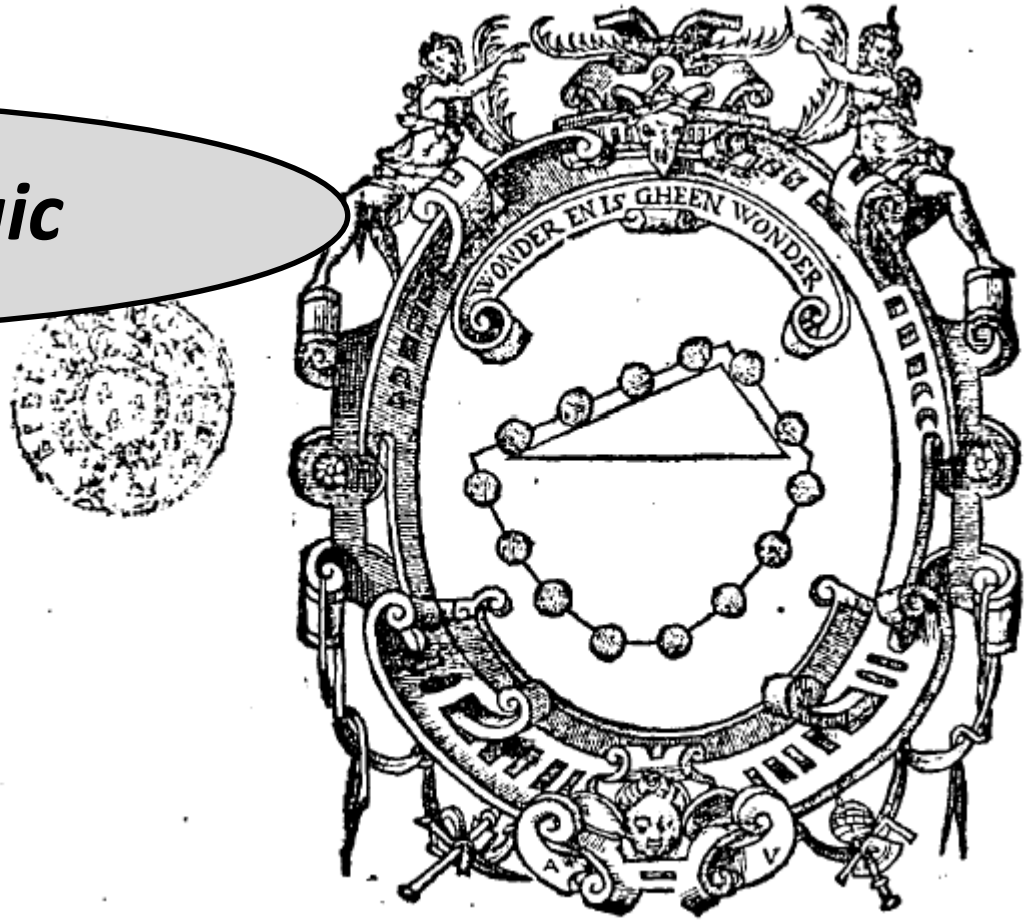
Conſcriptus à SIMONE STEVINO *Brugenſi.*

LVGODINI BATAVORVM,
Ex Officinâ Ioannis Patii, Academiæ Typographi.
Anno CIↃ IↃ CV.

STEVIN's "Epitath"

Discards Aristotle's marvel argument based upon movement and circular trajectories

Magic is no magic

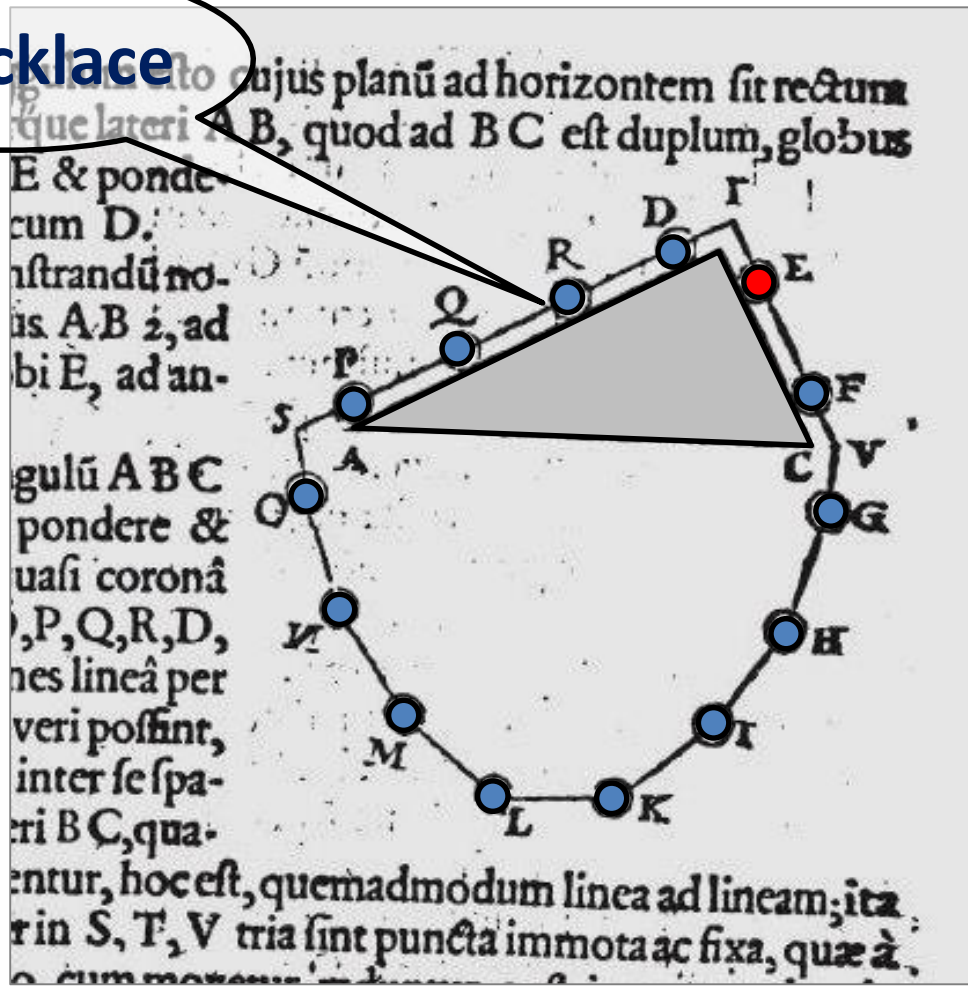


De Beghinselen der Weegconst (1586 /1

STEVIN's "Epitath"

The sloped plane

A necklace



De Beghinselen der Weegconst (1586 /1605)

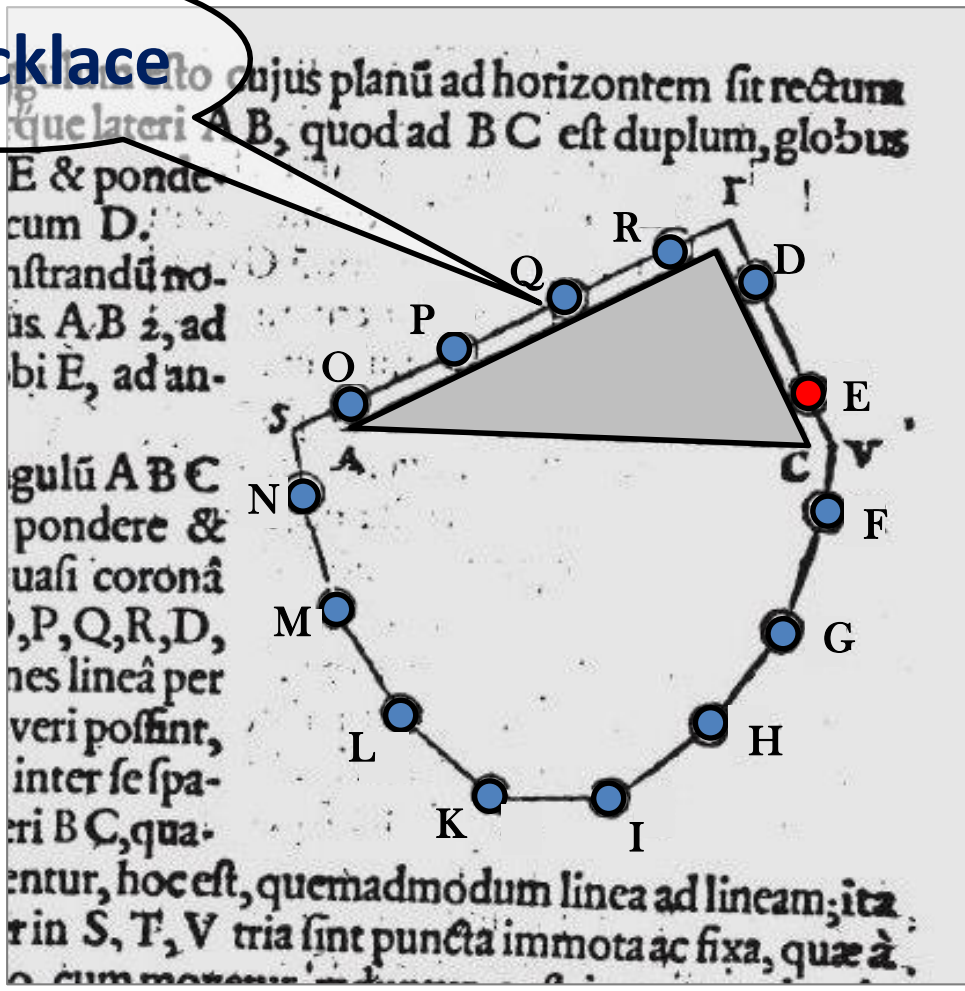
STEVIN's "Epitath"

The sloped plane

Impossibility of Perpetual motion

D & E equilibrate O, P, Q & R

A necklace



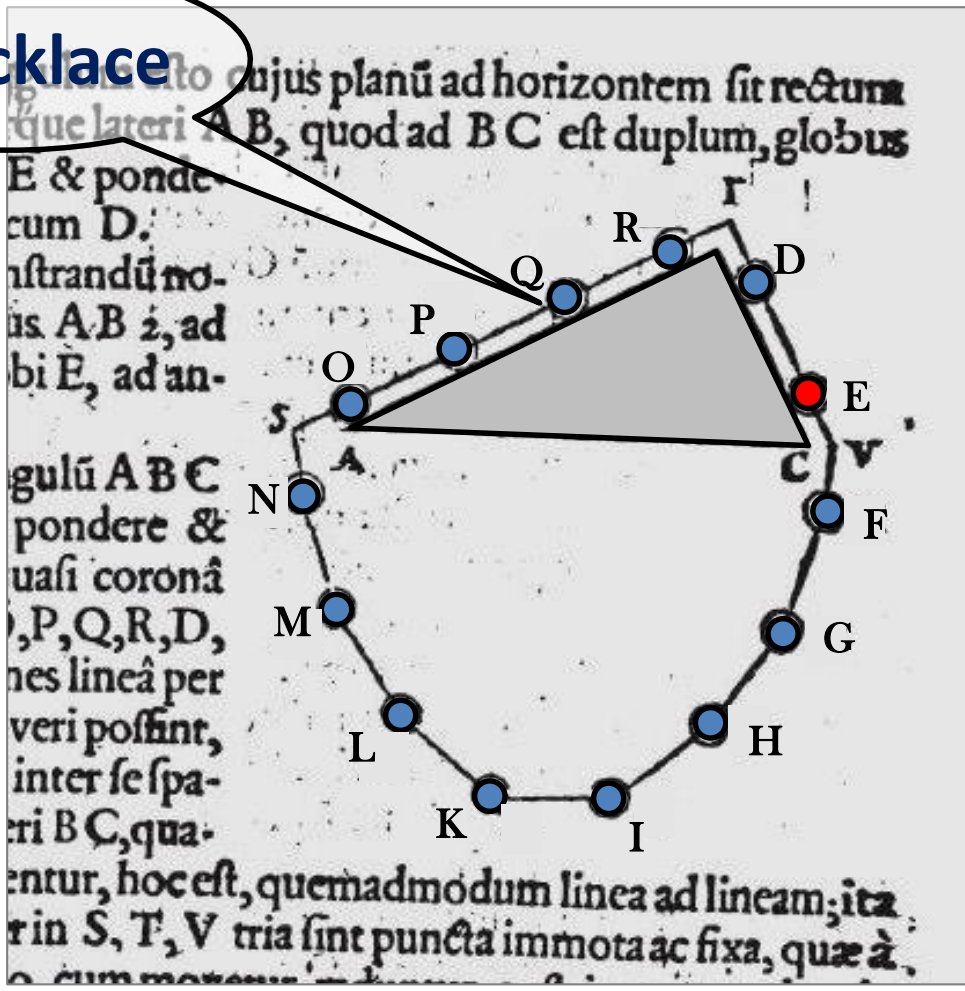
De Beghinselen der Weegconst (1586 /1605)

STEVIN's "Epitath"

The sloped plane

Kinematical thought experiment ?

A necklace

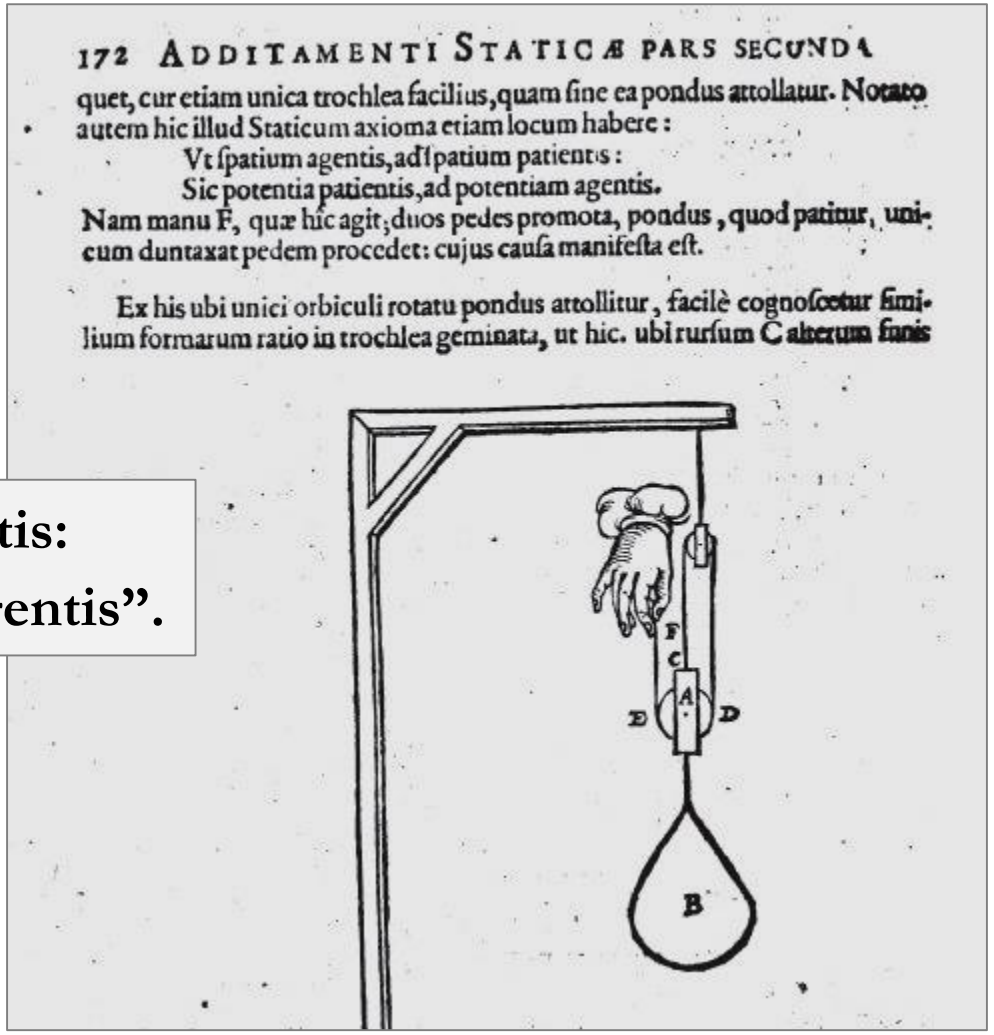


De Beghinselen der Weegconst (1586 /1605)

STEVIN

States a *Rule of proportion*

Pulleys & Pulley Blocks


172 ADDITAMENTI STATICÆ PARS SECUNDA

quet, cur etiam unica trochlea facilius, quam ſine ea pondus attollatur. Notato autem hic illud Staticum axioma etiam locum habere:

Vt ſpatium agentis, ad ſpatium patientis:
Sic potentia patientis, ad potentiam agentis.

Nam manu F, quæ hic agit; duos pedes promota, pondus, quod patitur, unicum duntaxat pedem procedet: cujus cauſa manifeſta eſt.

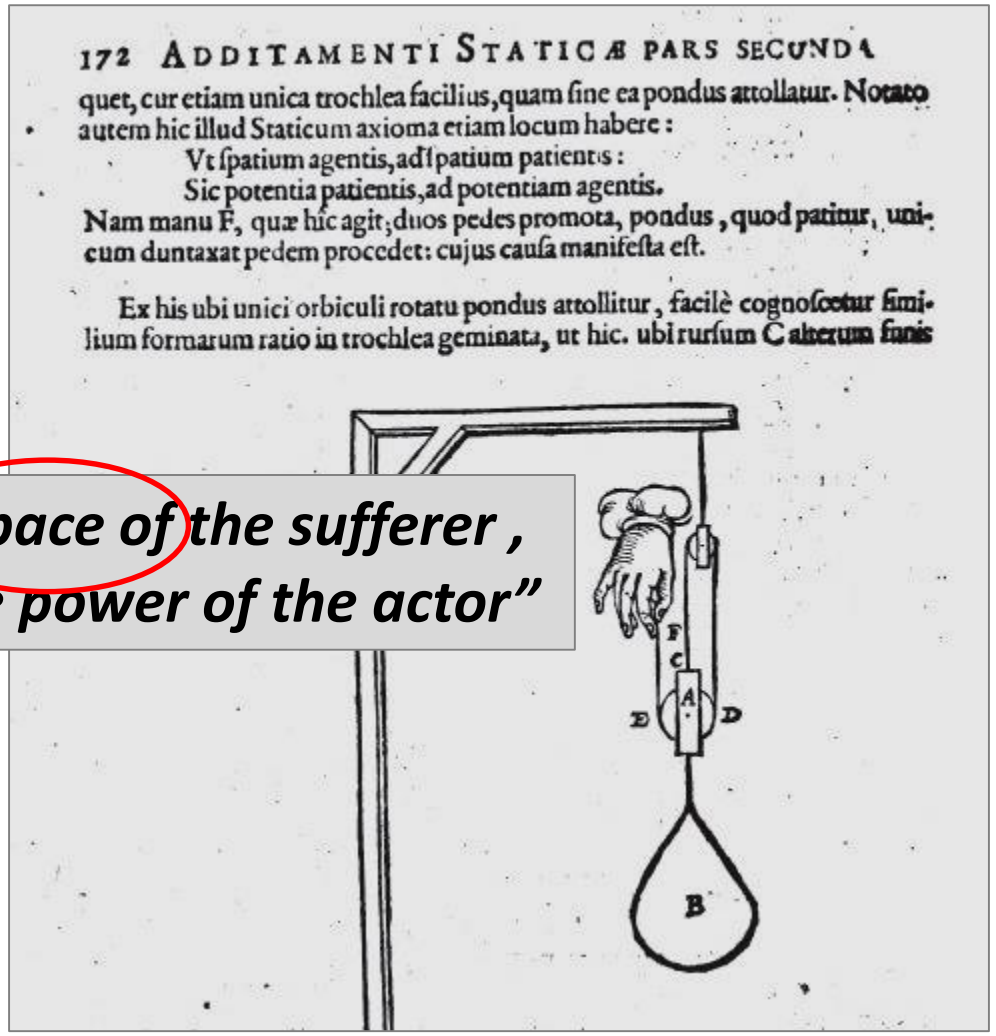
Ex his ubi unici orbiculi rotatu pondus attollitur, facilè cognoſcetur ſimilium formarum ratio in trochlea geminata, ut hic. ubi rurſum C alterum finis

"Ut spatium agentis ad spatium patientis:
Sic potentia patientis ad potentiam agentis".

STEVIN's criterion

States a *Rule of proportion* as a **criterion** not an explanation

Pulleys & Pulley Blocks


172 ADDITAMENTI STATICÆ PARS SECUNDA

quet, cur etiam unica trochlea facilius, quam ſine ea pondus attollatur. Notato autem hic illud Staticum axioma etiam locum habere:

Vt ſpatium agentis, ad ſpatium patientis:
Sic potentia patientis, ad potentiam agentis.

Nam manu F, quæ hic agit; duos pedes promota, pondus, quod patitur, unicum duntaxat pedem procedet: cujus cauſa manifeſta eſt.

Ex his ubi unici orbiculi rotatu pondus attollitur, facilè cognoſcetur ſimilium formarum ratio in trochlea geminata, ut hic. ubi rurſum C alterum funis

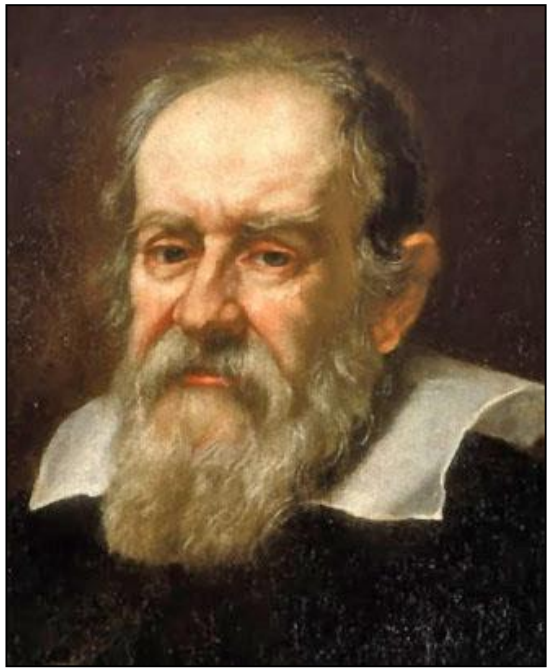
F C A E D B

"As the space of the actor is to the space of the sufferer ,
So is the power of the sufferer to the power of the actor"

(BENVENUTO's wording)

no time mentioned

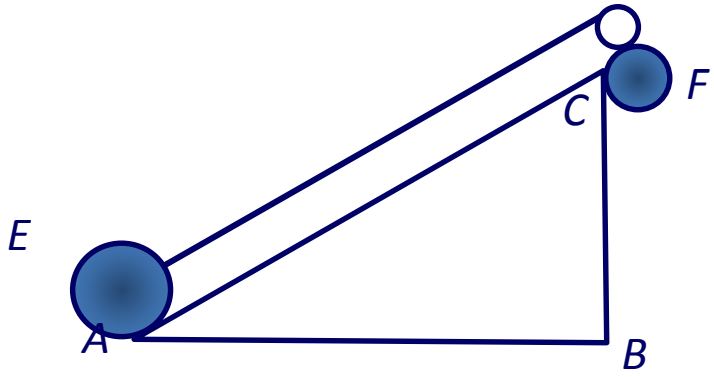
GALILEO (1564-1642)



Les Méchaniques (published in French,1634)
Discorsi… (1638)
Della Scienza meccanica (Ravena, 1649)

GALILEO

The sloped plane

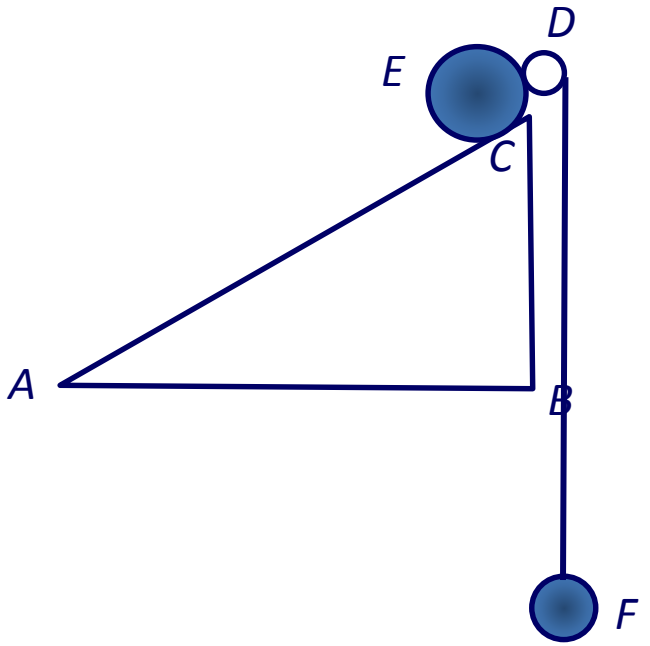


"…It is true that the body E will have covered all the line AC in the time the weight F falls down an equal length; but during this time, the body E will not have moved away from the common centre of weights more than the vertical length BC, while the weight F, falling down according to the vertical, has dropped a length equal to all the line AC."

Della Scienza meccanica (Ravena, 1649)

GALILEO

The sloped plane



BUT

" ...As a principle, we said that necessarily, in any mechanical instrument,
as much the force was increased
via this instrument,
as much, on the other hand,
one would lose time or velocity."

Della Scienza meccanica (Ravena, 1649)

LOOKING FOR A UNIFYING PRINCIPLE

DESCARTES (1596-1650)

DESCARTES

1637: a letter to Constantijn HUYGENS

Simple machines

Du 5 oct. 1637.

EXPLICATION

DES ENGINS PAR L'AYDE DESQVELS ON PEVT AVEC VNE PETITE FORCE LEVER VN FARDEAV FORT PESANT.

L'inuention de tous ces engins n'eſt fondée que ſur vn ſeul principe, qui eſt que la meſme force qui peut leuer vn poids, par exemple, de cent liures a la hau-

teur de deux pieds, en peut auſſy leuer vn de 200 liures, a la hauteur d'vn pied, ou vn de 400 a la hauteur d'vn demi pied, & ainſy des autres, ſi tant eſt qu'elle luy ſoit appliquée.

Et ce principe ne peut manquer d'eſtre receu, ſi on conſidere que l'effect doit eſtre touſiours proportionné a l'action qui eſt neceſſaire pour le produire :

DESCARTES

1637: a letter to Constantijn HUYGENS

"The invention of all these machines is founded on one principle, which is that the same force which can lift a weight, for example of 100 pounds, up to two feet, can also lift a weight of 200 pounds up to one foot, or a weight of 400 pounds up to half a foot..."

DESCARTES

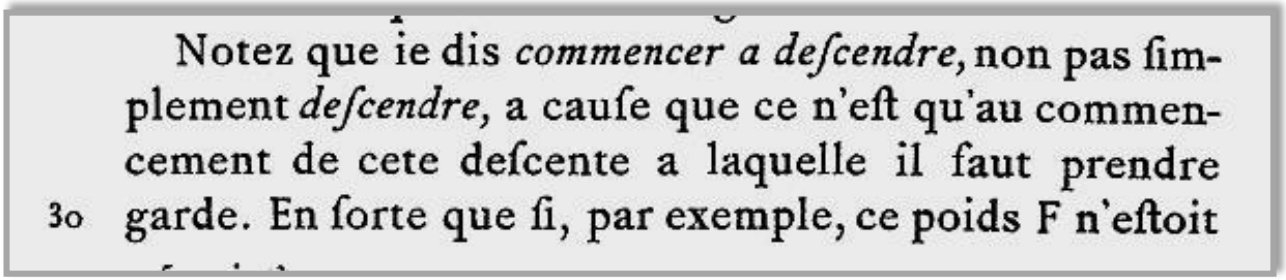
1638: a letter to MERSENNE

"From this it follows evidently that the gravity relative to a given body, or equivalently the force to be exerted to sustain it or prevent it from going down, when it is in a given position, should be measured by means of the beginning of the movement that would be done by the power which sustains it either for lifting it or following it if it went down."

Opens the way to the concept of VIRTUALITY

DESCARTES

1638: a letter to MERSENNE



Notez que ie dis *commencer a deſcendre*, non pas ſimplement *deſcendre*, a cauſe que ce n'eſt qu'au commencement de cete deſcente a laquelle il faut prendre garde. En ſorte que ſi, par exemple, ce poids F n'eſtoit

"…Note that I say begin to go down and not simply go down, because it is only the beginning of the descent that must be taken into account"

Opens the way to the concept of VIRTUALITY

DESCARTES

1646: a letter to BOSWELL

*"I do not deny the material truth of what Mechanicists usually say, namely that the higher the velocity of the longer arm of the lever compared with the shorter arm, the smaller the force necessary to move it; **but I do deny that velocity or slowness be the cause of this effect**."*

Opens the way to the concept of VIRTUALITY

DESCARTES

1646: a letter to BOSWELL

"I do r
Mecho
higher the velocity of the longer arm of the lever compared with the shorter arm, the smalle
do de
of this

Referring to velocities is irrelevant

Incipient movement. No time involved

Opens the way to the concept of VIRTUALITY

The AMBIGUITY introduced by ARISTOTLE is finally RULED OUT

Johann BERNOULLI

1717: a letter to VARIGNON

Relates equilibrium to motion through

- ***A definition of VIRTUAL VELOCITIES***
- ***A definition of ENERGY***
- ***A 1st statement of***

the PRINCIPLE of VIRTUAL VELOCITIES

Johann BERNOULLI

1717: a letter to VARIGNON

- ***Definition of VIRTUAL VELOCITIES***

> ***"… these movements forward or backward, which are what I call virtual velocities, are just what the quantities in which each tendency line increases or decreases in the small movement."***

Component of the small displacement along the line of action of the force

Johann BERNOULLI

1717: a letter to VARIGNON

Despite the terminology: NO TIME involved

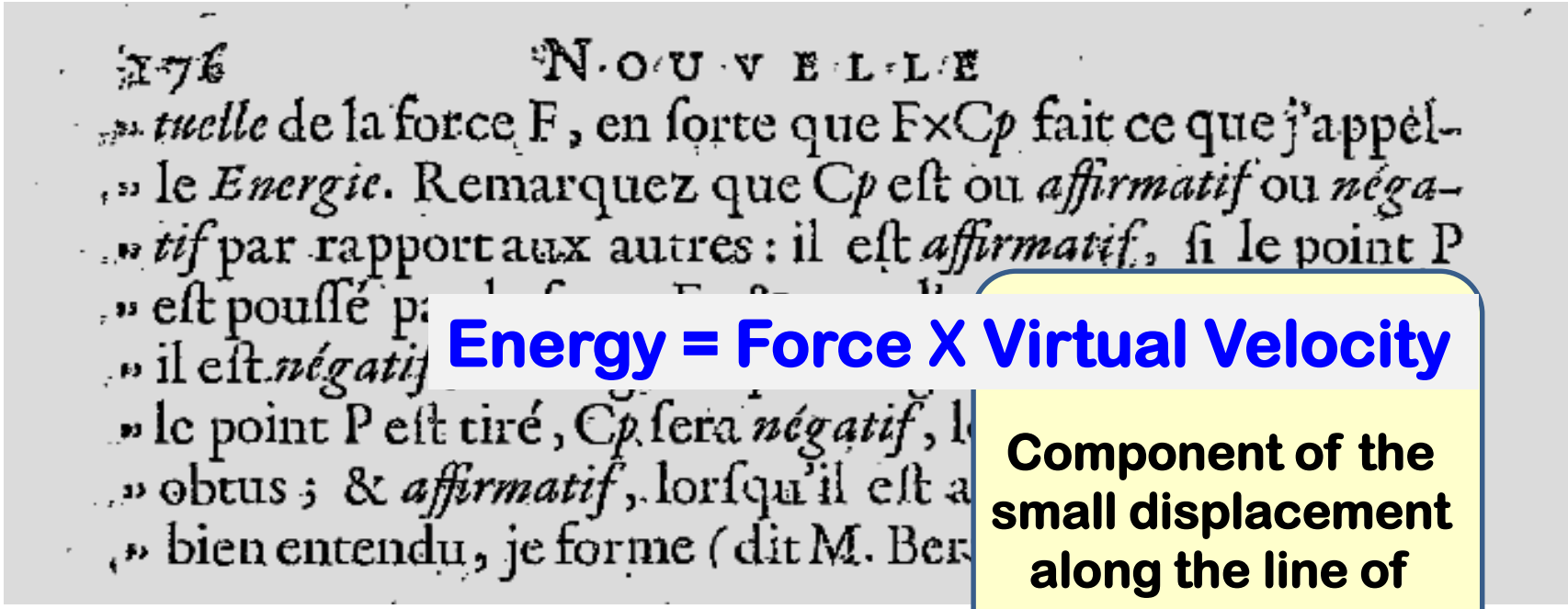
"… these movements forward or backward, which are what I call virtual velocities, are just what the quantities in which each tendency line increases or decreases in the small movement."

Component of the small displacement along the line of action of the force

Johann BERNOULLI

1717: a letter to VARIGNON

- ***Definition of ENERGY***



176 NOUVELLE

» *tuelle* de la force F, en ſorte que F×*Cp* fait ce que j'appel-
» le *Energie*. Remarquez que *Cp* eſt ou *affirmatif* ou *néga-*
» *tif* par rapport aux autres : il eſt *affirmatif*, ſi le point P
» eſt pouſſé p
» il eſt *négati*
» le point P eſt tiré, *Cp* ſera *négatif*, l
» obtus ; & *affirmatif*, lorſqu'il eſt a
» bien entendu, je forme (dit M. Ber

Energy = Force X Virtual Velocity

Component of the small displacement along the line of action of the force

Johann BERNOULLI

1717: a letter to VARIGNON

- ***A 1st statement of the PRINCIPLE of VIRTUAL VELOCITIES***

> ***"For any equilibrated system of forces…the sum of the affirmative energies will be equal to the sum of negative energies counted positive"***

NO TIME involved

Johann BERNOULLI's Principle

EQUILIBRIUM is analysed through
KINEMATICAL THOUGHT EXPERIMENTS

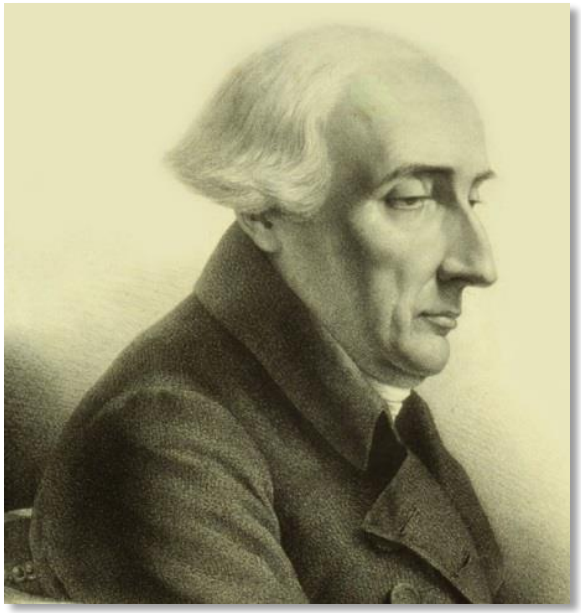
"For any equilibrated system of forces…the sum of the affirmative energies will be equal to the sum of negative energies counted positive"
[in any small rigid body motion]

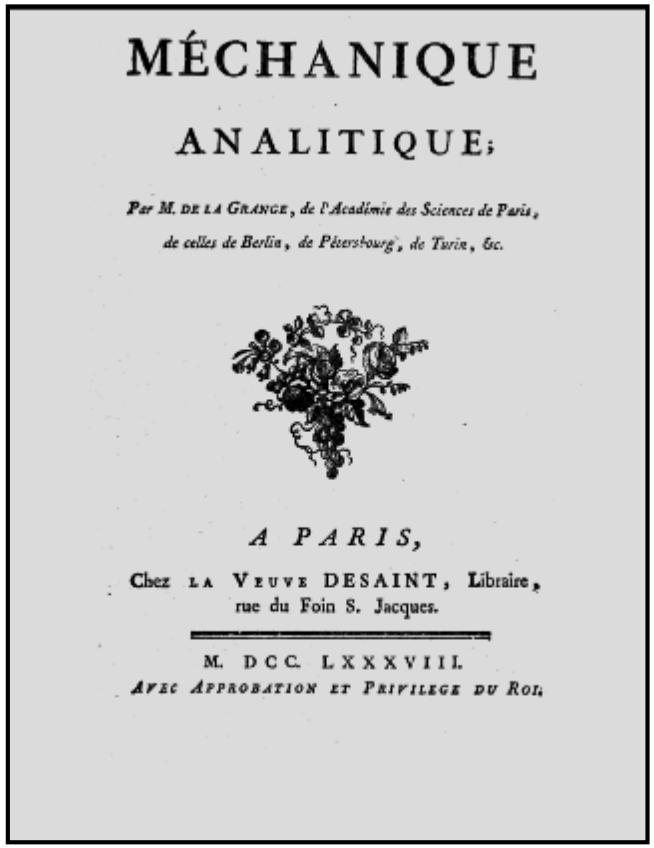
NO TIME involved

The PRINCIPLE of VIRTUAL VELOCITIES

to

LAGRANGE (1736-1813)





MÉCHANIQUE

ANALITIQUE;

Par M. de la Grange, de l'Académie des Sciences de Paris, de celles de Berlin, de Pétersbourg, de Turin, &c.

A PARIS,

Chez la Veuve DESAINT, Libraire, rue du Foin S. Jacques.

M. DCC. LXXXVIII.

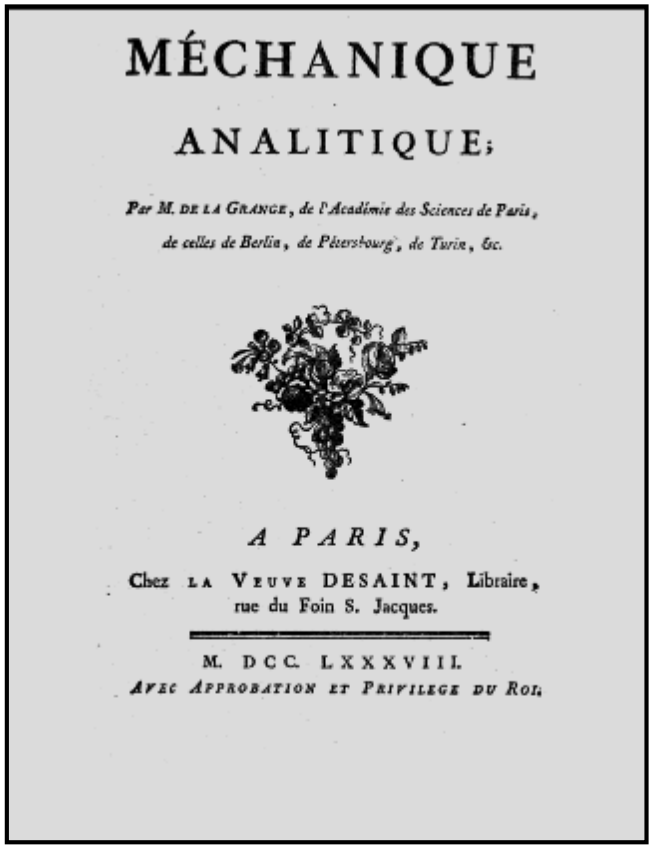
Avec Approbation et Privilege du Roi.

1st edition, 1788

LAGRANGE

- ***Defines FORCES***
- ***Defines VIRTUAL VELOCITIES***
- ***States the***

  PRINCIPLE of VIRTUAL VELOCITIES



MÉCHANIQUE

ANALITIQUE;

Par M. de la Grange, de l'Académie des Sciences de Paris, de celles de Berlin, de Pétersbourg, de Turin, &c.

A PARIS,

Chez la Veuve DESAINT, Libraire, rue du Foin S. Jacques.

M. DCC. LXXXVIII.

Avec Approbation et Privilege du Roi.

1st edition, 1788

LAGRANGE

***Méchanique Analitique*, 1st edition, 1788**

- ***Defines FORCES***

> ***"We generally mean by force or power*** **[puissance]** ***the cause, whatever it is, which imparts or tends to impart a movement to the body to which it is supposed to be applied."***

LAGRANGE

***Méchanique Analitique*, 1st edition, 1788**

- ***Defines VIRTUAL VELOCITIES***

cf. **DESCARTES**

"One must understand by virtual velocity, the velocity which a body in equilibrium would be ready to receive, in case this equilibrium should be upset; i.e. the velocity that this body would really take in the first instant of its movement."

LAGRANGE

***Méchanique Analitique*, 1st edition, 1788**

- ***The PRINCIPLE of VIRTUAL VELOCITIES***

> ***"But this principle is not only very simple and very general in itself; as an invaluable and unique advantage it can also be expressed in a general formula which encompasses all the problems that can be proposed regarding equilibrium."***

ENTHUSIASTICALLY

LAGRANGE

***Méchanique Analitique*, subsequent editions**

- ***The PRINCIPLE of VIRTUAL VELOCITIES***

  "18. Regarding the nature of the principle of virtual velocities, it must be recognized that it is not self-evident enough to be settled as a primitive principle"

LAGRANGE

Generalises BERNOULLI's Statement of the principle
to a system of bodies
and unspecified small movements

Instead of the classical reference to the lever,
provides his own "Proof" of the principle based
upon the equilibrium of a system of pulley blocks

***Journal de l'école polytechnique*, 1797**

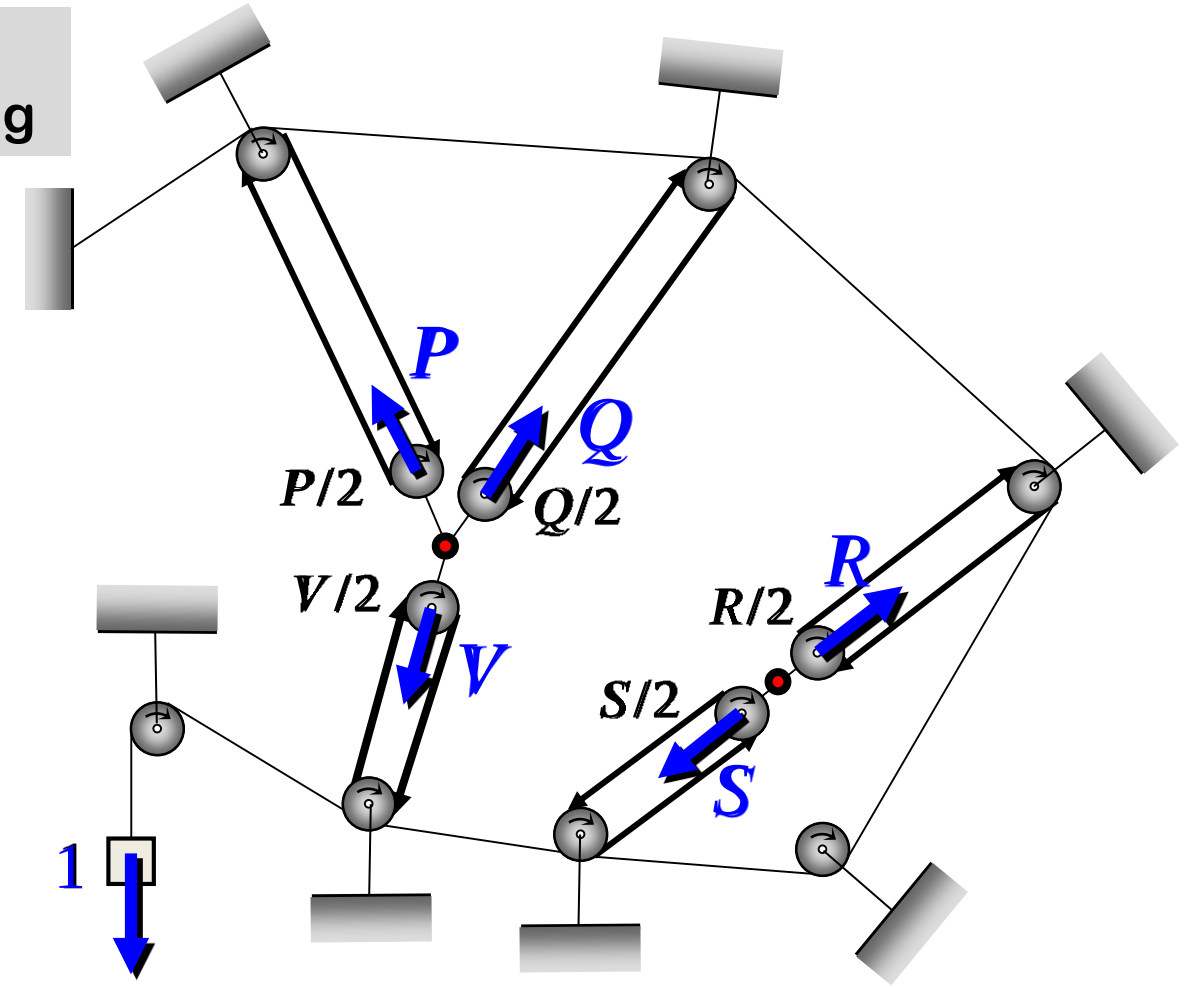
LAGRANGE

"You will not find Figures in this Work. The methods I use require neither constructions nor geometrical or mechanical arguments, but only algebraic operations, in a regular and uniform course."

LAGRANGE's "Proof" of the Principle

ideally inextensible,
flexible, weightless string

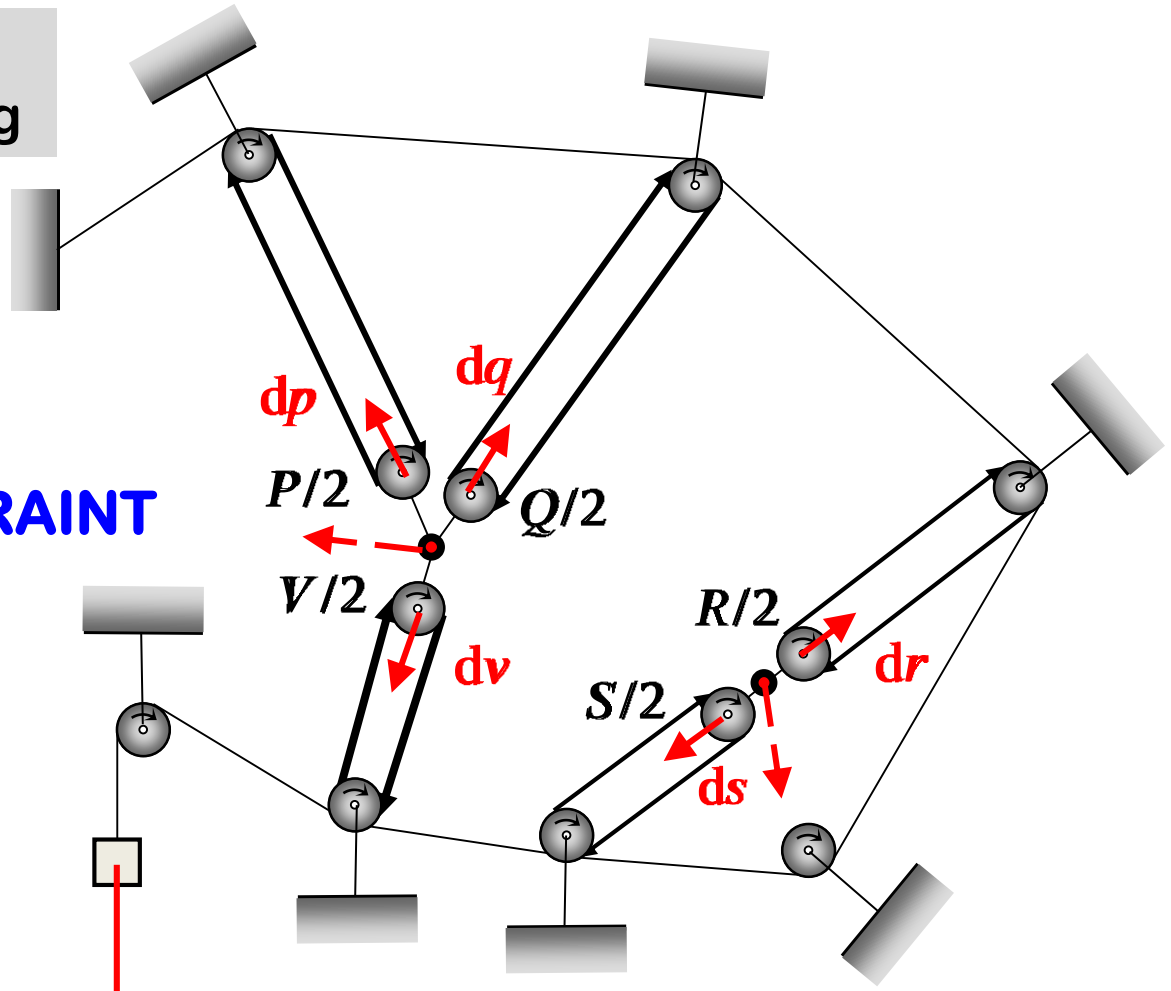
Pulley Principle



LAGRANGE's "Proof" of the Principle

ideally inextensible,
flexible, weightless string

NO
GEOMETRICAL CONSTRAINT

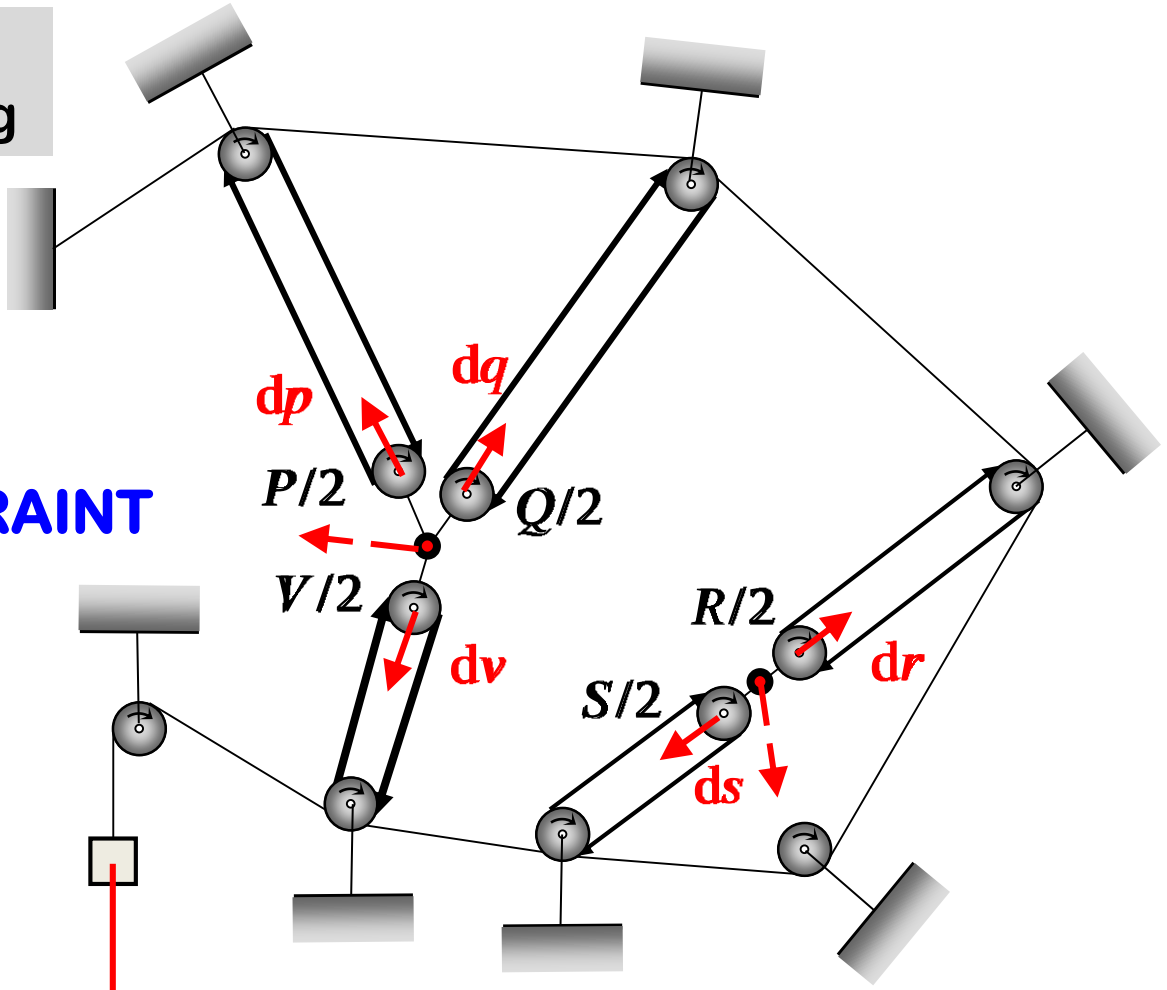


$$P\mathrm{d}p + Q\mathrm{d}q + V\mathrm{d}v + R\mathrm{d}r + S\mathrm{d}s$$

LAGRANGE's "Proof" of the Principle

ideally inextensible,
flexible, weightless string

EQUILIBRIUM
NO
GEOMETRICAL CONSTRAINT

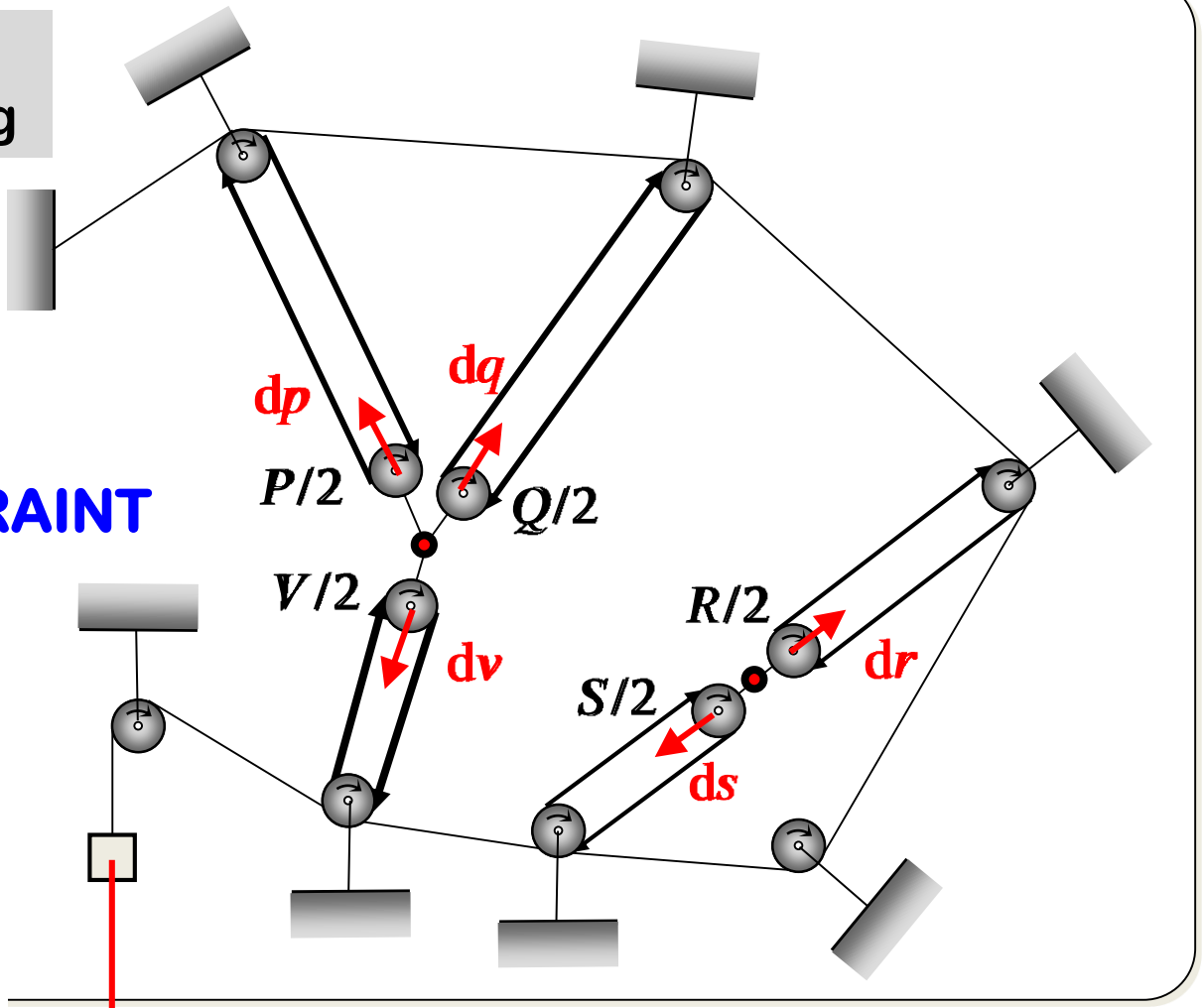


$$P\mathrm{d}p + Q\mathrm{d}q + V\mathrm{d}v + R\mathrm{d}r + S\mathrm{d}s = 0$$

LAGRANGE's "Proof" of the Principle

ideally inextensible,
flexible, weightless string

EQUILIBRIUM
NO
GEOMETRICAL CONSTRAINT



Just the Principle of Virtual Velocities !

$$P\mathrm{d}p + Q\mathrm{d}q + V\mathrm{d}v + R\mathrm{d}r + S\mathrm{d}s = 0$$

FORCES defined through DUALITY

Analytical expression of the Principle

$$P\,\mathrm{d}p + Q\,\mathrm{d}q + R\,\mathrm{d}r + \ldots = 0$$

$$\forall \mathrm{d}x_i', \forall \mathrm{d}x_i'', \ldots$$

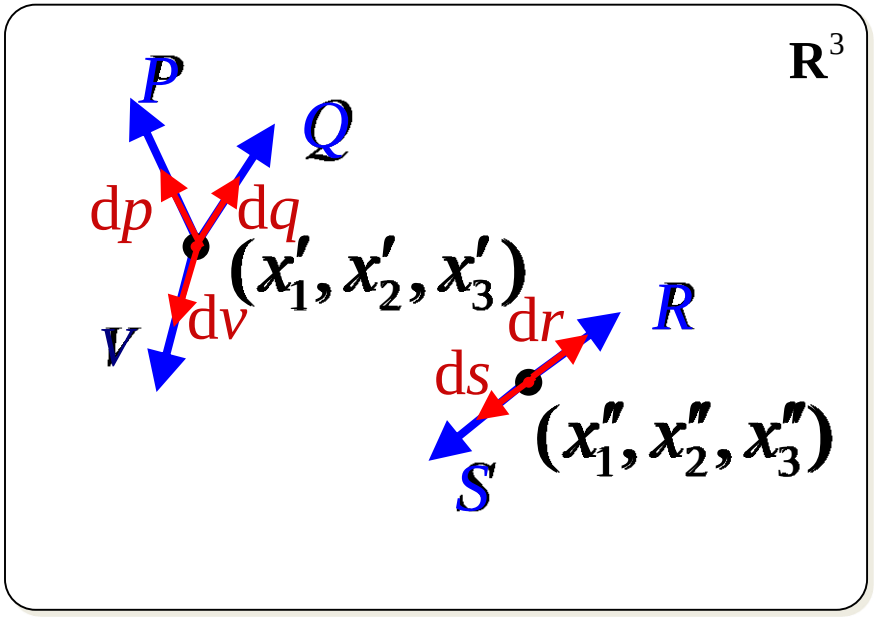
No

geometrical constraint

$$\mathrm{d}p = \frac{\partial p}{\partial x_i'}\mathrm{d}x_i' + \frac{\partial p}{\partial x_i''}\mathrm{d}x_i'' \quad i = 1,2,3$$

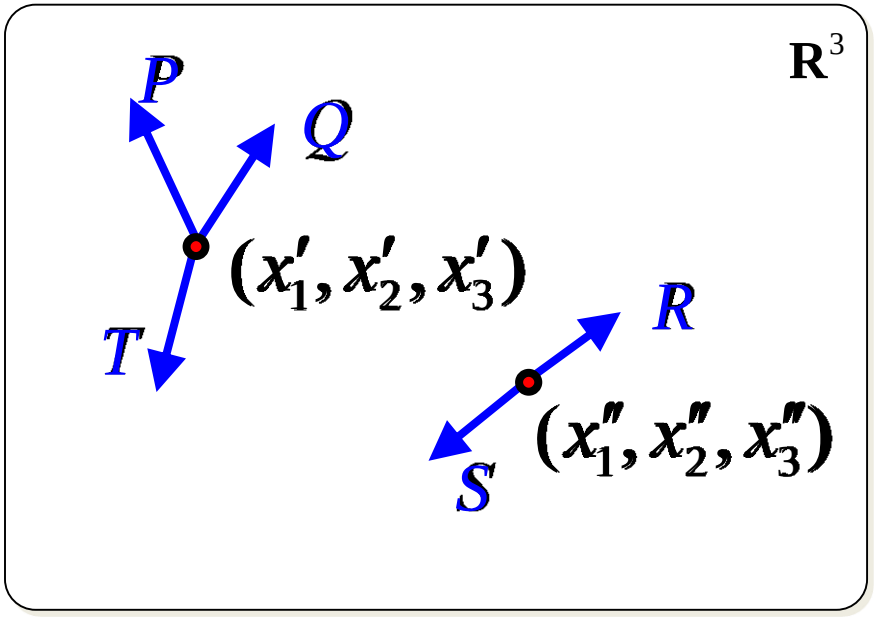
$$\mathrm{d}q = \frac{\partial q}{\partial x_i'}\mathrm{d}x_i' + \frac{\partial q}{\partial x_i''}\mathrm{d}x_i'', \ldots \quad i = 1,2,3$$

$$\mathrm{d}r = \frac{\partial r}{\partial x_i'}\mathrm{d}x_i' + \frac{\partial r}{\partial x_i''}\mathrm{d}x_i'', \ldots \quad i = 1,2,3$$



Geometrical constraints

WHAT ABOUT GEOMETRICAL CONSTRAINTS



Geometrical constraints

$$P\,\mathrm{d}p + Q\,\mathrm{d}q + R\,\mathrm{d}r + \ldots = 0$$

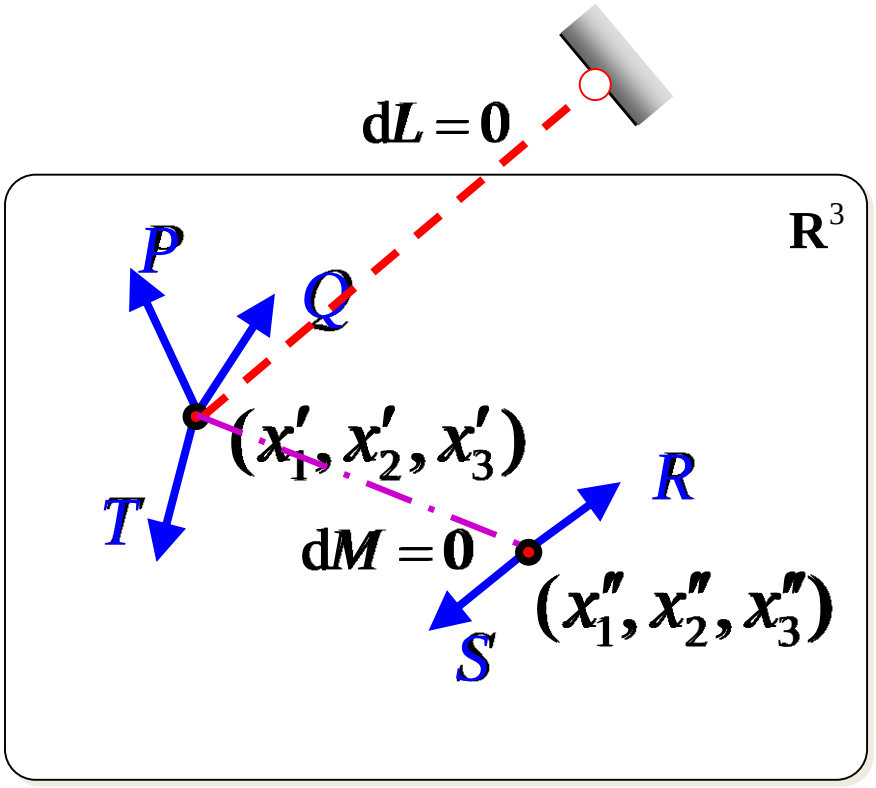
$\forall \mathrm{d}x_i', \forall \mathrm{d}x_i'', \ldots$ such that

$\mathrm{d}L = 0,\ \mathrm{d}M = 0$

External or Internal

Differential Constraint Equations

$$\begin{cases} \mathrm{d}L = \dfrac{\partial L}{\partial x_i'}\mathrm{d}x_i' + \dfrac{\partial L}{\partial x_i''}\mathrm{d}x_i'' + \ldots = 0 \\ \mathrm{d}M = \dfrac{\partial M}{\partial x_i'}\mathrm{d}x_i' + \dfrac{\partial M}{\partial x_i''}\mathrm{d}x_i'' + \ldots = 0 \end{cases}$$



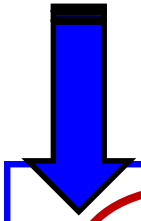
Geometrical constraints

$$P\,\mathrm{d}p + Q\,\mathrm{d}q + R\,\mathrm{d}r + \ldots = 0$$

$\forall \mathrm{d}x_i', \forall \mathrm{d}x_i'', \ldots$ such that

$\mathrm{d}L = 0,\ \mathrm{d}M = 0$

Theory of linear equations



$\exists \lambda, \mu,$ such that $\forall \mathrm{d}x_i', \forall \mathrm{d}x_i'', \ldots$

$$P\,\mathrm{d}p + Q\,\mathrm{d}q + R\,\mathrm{d}r + \ldots + \lambda\,\mathrm{d}L + \mu\,\mathrm{d}M = 0$$

LAGRANGE Multipliers

$$\exists \lambda, \mu, \text{ such that } \forall \mathrm{d}x_i', \forall \mathrm{d}x_i'', \ldots$$

$$P\,\mathrm{d}p + Q\,\mathrm{d}q + R\,\mathrm{d}r + \ldots + \lambda\,\mathrm{d}L + \mu\,\mathrm{d}M = 0$$

Crucial step forward comes from the mathematical similarity

between $P\,\mathrm{d}p + Q\,\mathrm{d}q + R\,\mathrm{d}r + \ldots$

and $\lambda\,\mathrm{d}L + \mu\,\mathrm{d}M + \ldots$

LAGRANGE Multipliers

$$\exists \lambda, \mu, \text{ such that } \forall \mathrm{d}x_i', \forall \mathrm{d}x_i'', \ldots$$
$$P\,\mathrm{d}p + Q\,\mathrm{d}q + R\,\mathrm{d}r + \ldots + \lambda\,\mathrm{d}L + \mu\,\mathrm{d}M = 0$$

Crucial step forward comes from the mathematical similarity

"7. It comes out then that each geometrical constraint equation is equivalent to one or several forces acting on the system, along given directions or, as a general rule, tending to vary the values of the given functions; so that the same state of equilibrium will be obtained for the system, either using these forces or the constraint equations."

LAGRANGE Multipliers

In proper words, these forces stand as the resistances that the bodies should meet for being linked to each other or due to the obstacles that may impede their motion; or rather, these forces are precisely the resistances, which must be equal and opposite to the pressures exerted by the bodies."

BINDING and INTERNAL FORCES
are defined from the given geometrical constraints,
either external or internal,
through the concept of DUALITY
on the VIRTUAL VELOCITIES

LAGRANGE Multipliers

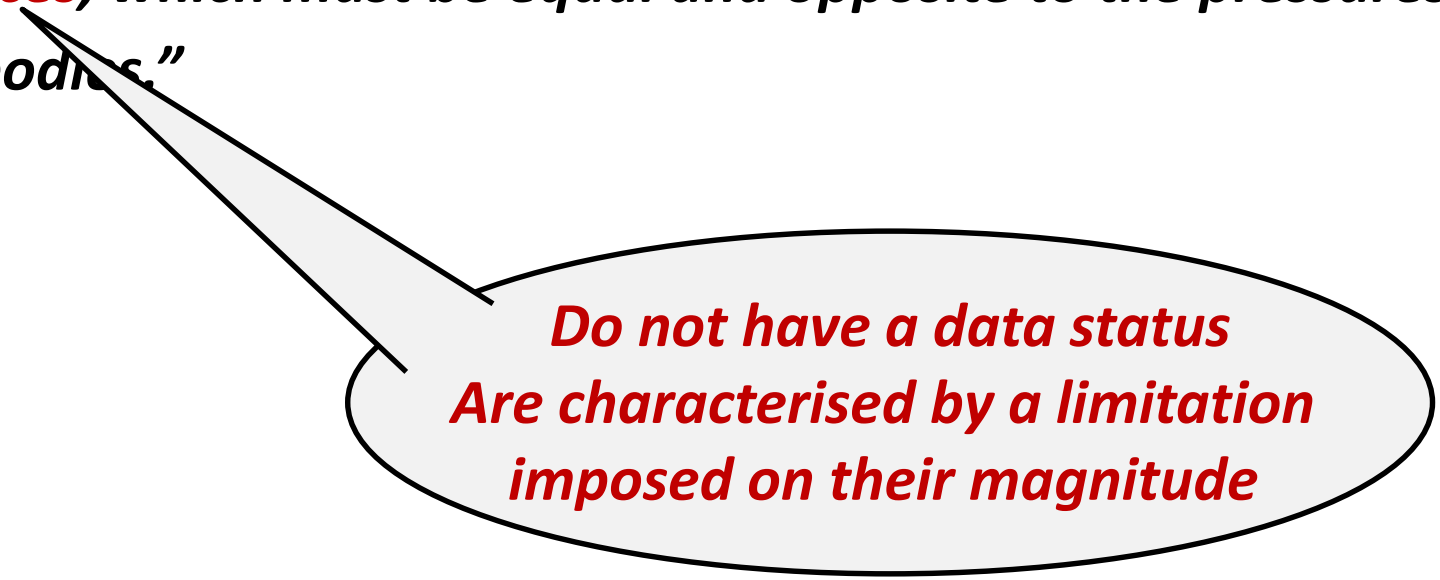
In proper words, these forces stand as the resistances that the bodies should meet for being linked to each other or due to the obstacles that may impede their motion; or rather, these forces are precisely the resistances, which must be equal and opposite to the pressures exerted by the bodies."

To be compared with

"We generally mean by force or power [puissance] *the cause, whatever it is, which imparts or tends to impart a movement to the body to which it is supposed to be applied."*

LAGRANGE Multipliers

In proper words, these forces stand as the resistances that the bodies should meet for being linked to each other or due to the obstacles that may impede their motion; or rather, these forces are precisely the resistances, which must be equal and opposite to the pressures exerted by the bodies."

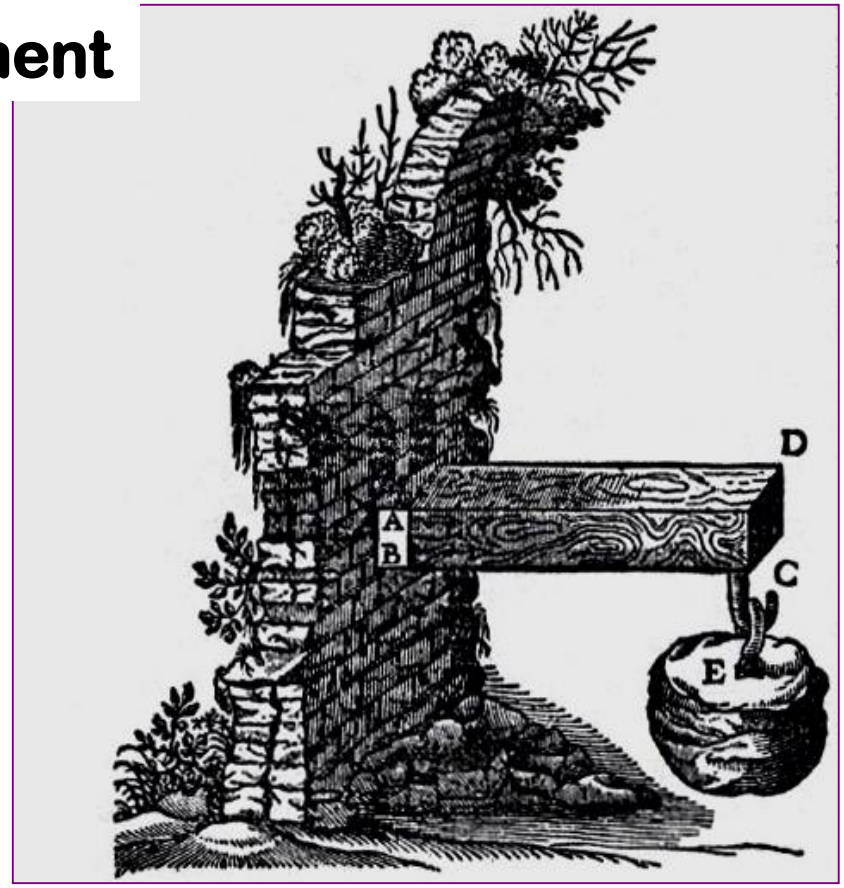


Back to

GALILEO

Looking for the bearing capacity of the cantilever beam

Kinematical Thought Experiment

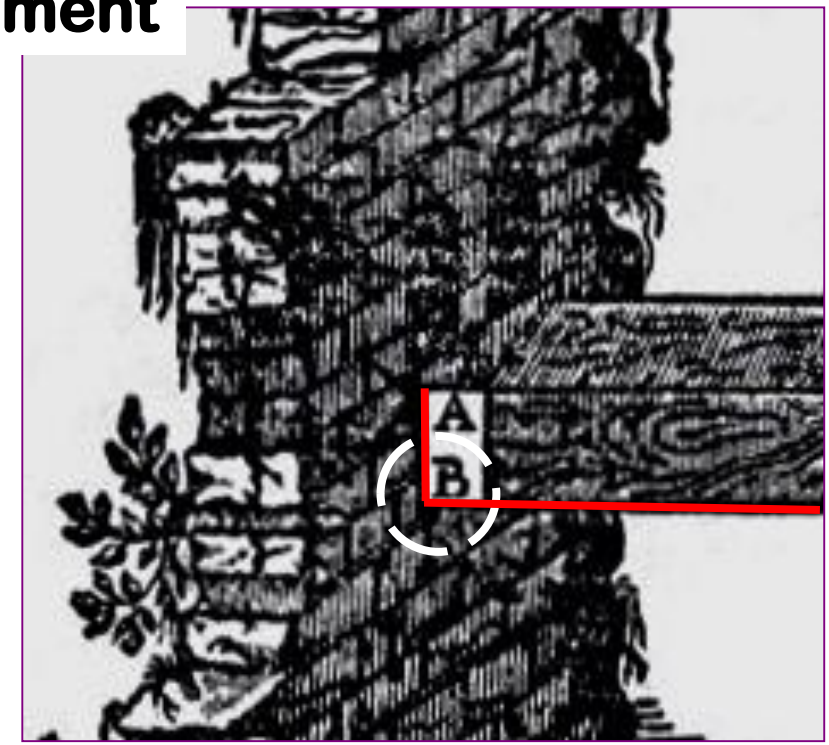


Back to

GALILEO

Looking for the bearing capacity of the cantilever beam

Kinematical Thought Experiment



"…It is clear that, if the cylinder breaks, fracture will occur at the point B where the edge of the mortise acts as a fulcrum for the lever BC, to which the force is applied; the thickness of the solid BA is the other arm of the lever along which is located the resistance…"

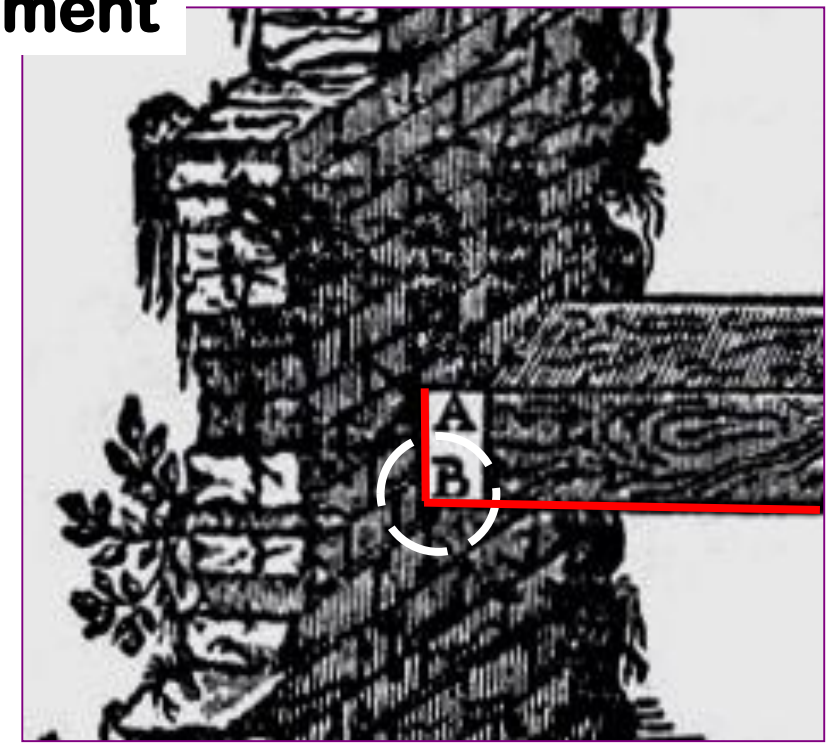
Back to

GALILEO

Looking for the bearing capacity of the cantilever beam

Kinematical Thought Experiment

Potential rigid body rotation
about point B
Beam treated as a
rectangular lever.



Virtual motion
Resistance defined through Duality

DYNAMICS ?

What about Dynamics?

NEWTON's 2nd Law

In a Galilean Reference Frame

$$\underline{F} = m\,\underline{a}$$

2nd Law.

The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed

Isaac Newton, *The Principia*, translation by Andrew Motte, 1729.

What about Dynamics?

NEWTON's 2nd Law

In a Galilean Reference Frame

$$\underline{F} = m\,\underline{a}$$

D'ALEMBERT's Principle

In a Galilean Reference Frame

$$\underline{F} - m\,\underline{a} = 0$$

$\underline{F}$ **and** $(-m\,\underline{a})$

are in equilibrium

Will be treated the same way in the Principle of virtual velocities

CityU 香港城市大學 City University of Hong Kong Professional•Creative For The World

City University of Hong Kong Institute for Advanced Study

THANK YOU

FOR YOUR PRECIOUS TIME

jean.salencon@m4x.org